On exceptional points for the Lebesgue density theorem

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Based upon joint work with R. Peirone.

Let m_d denote the Lebesgue measure on $\mathbb{R}$, $B \subset \mathbb{R}$ a Borel set, and χ_B the characteristic function of B. For every norm $\| \cdot \|$ on $\mathbb{R}^d$ and every $x \in \mathbb{R}^d$, we denote

$$B_{\| \cdot \|}(x, r) := \{y \in \mathbb{R}^d ; \|y - x\| \leq r\}, \qquad r > 0,$$

$$\underline{D}_{\| \cdot \|, B}(x) := \liminf_{0<r\to 0} \frac{m_d\Big(B \cap B_{\| \cdot \|}(x, r)\Big)}{m_d\Big(B_{\| \cdot \|}(x, r)\Big)},$$

$$\overline{D}_{\| \cdot \|, B}(x) := \limsup_{0<r\to 0} \frac{m_d\Big(B \cap B_{\| \cdot \|}(x, r)\Big)}{m_d\Big(B_{\| \cdot \|}(x, r)\Big)}.$$

If $\underline{D}_{\| \cdot \|, B}(x) = \overline{D}_{\| \cdot \|, B}(x)$, then we denote this common value

$$\lim_{0<r\to 0} \frac{m_d\Big(B \cap B_{\| \cdot \|}(x, r)\Big)}{m_d\Big(B_{\| \cdot \|}(x, r)\Big)} \tag{1}$$

by $D_{\| \cdot \|, B}(x)$ and call it *density of B at x*. If even the limit

$$\lim_{\substack{0 < r \to 0 \\ x \in B_{\| \cdot \|}(y, r)}} \frac{m_d\Big(B \cap B_{\| \cdot \|}(y, r)\Big)}{m_d\Big(B_{\| \cdot \|}(y, r)\Big)} \tag{2}$$

exists, then we say that the density of B at x exists in *strong sense*.

We notice that, in the one-dimensional case $d = 1$, the existence of the density of a Borel set $B \subset \mathbb{R}$ at $x \in \mathbb{R}$ in the strong sense (2) means the derivability of the function

$$f_a : (a, +\infty) \ni t \longmapsto m_1\Big(B \cap (a, t)\Big)$$

at x (where $a < x$ can be chosen arbitrarily), while the existence of the density of B at x in the ordinary sense (1) means the symmetric derivability of the above f_a at x.

By the Lebesgue density theorem we have for every norm $\|\cdot\|$ on $\mathbb{R}^d$ and m_d-almost every $x \in \mathbb{R}^d$:

$$D_{\|\cdot\|,B}(x) = \underline{D}_{\|\cdot\|,B}(x) = \overline{D}_{\|\cdot\|,B}(x) = \chi_B(x).$$

In particular, if we assume that $m_d(B) > 0$ and $m_d(\mathbb{R}^d \setminus B) > 0$, then the functions $\underline{D}_{\|\cdot\|,B}$ and $\overline{D}_{\|\cdot\|,B}$ take both values 1 and 0.

Points with $D_{\|\cdot\|,B}(x) = 1$ or 0

Let $B \subset \mathbb{R}^d$ be a Borel set, and $x \in \mathbb{R}^d$. If the density $D_{\|\cdot\|,B}(x)$ exists and is equal to 0 for some norm $\|\cdot\|$ on $\mathbb{R}^d$, then, for every norm $|||\cdot|||$ on $\mathbb{R}^d$, the limit

$$\lim_{\substack{0<r\to 0\\ x\in B_{|||\cdot|||}(y,r)}} \frac{m_d\Big(B\cap B_{|||\cdot|||}(y,r)\Big)}{m_d\Big(B_{|||\cdot|||}(y,r)\Big)}$$

exists and is equal to 0.

Proof. First of all we notice that, since any two norms on the finite-dimensional vector space $\mathbb{R}^d$ are equivalent, there exist constants $0 < c_1 < c_2$ such that

$$c_1\,\|y\| \le |||y||| \le c_2\,\|y\|\,, \qquad y \in \mathbb{R}^d.$$

Thus, for every $y \in \mathbb{R}^d$ with $x \in B_{|||\cdot|||}(y,r)$,

$$\begin{aligned} B_{\|\cdot\|}\Big(y,\frac{r}{c_2}\Big) &\subset B_{|||\cdot|||}(y,r) \\ &\subset B_{|||\cdot|||}(x,2r) \subset B_{\|\cdot\|}\Big(x,\frac{2r}{c_1}\Big). \end{aligned}$$

Consequently:

$$\frac{m_d\Big(B \cap B_{|||\cdot|||}(y,r)\Big)}{m_d\Big(B_{|||\cdot|||}(y,r)\Big)}$$

$$\leq \frac{m_d\Big(B \cap B_{\|\cdot\|}(x,\frac{2r}{c_1})\Big)}{m_d\Big(B_{\|\cdot\|}(y,\frac{2r}{c_2})\Big)}$$

$$= \underbrace{\frac{m_d\Big(B_{\|\cdot\|}(x,\frac{2r}{c_1})\Big)}{m_d\Big(B_{\|\cdot\|}(y,\frac{2r}{c_2})\Big)}}_{=\left(\frac{2r}{c_1}\cdot\frac{c_2}{2r}\right)^d=\left(\frac{c_2}{c_1}\right)^d} \cdot \frac{m_d\Big(B \cap B_{\|\cdot\|}(x,\frac{2r}{c_1})\Big)}{m_d\Big(B_{\|\cdot\|}(x,\frac{2r}{c_1})\Big)}$$

$$= \left(\frac{c_2}{c_1}\right)^d \cdot \frac{m_d\Big(B \cap B_{\|\cdot\|}(x,\frac{2r}{c_1})\Big)}{m_d\Big(B_{\|\cdot\|}(x,\frac{2r}{c_1})\Big)}.$$

We conclude:

$$\limsup_{\substack{0<r\to 0\\ x\in B_{|||\cdot|||}(y,r)}} \frac{m_d\Big(B \cap B_{|||\cdot|||}(y,r)\Big)}{m_d\Big(B_{|||\cdot|||}(y,r)\Big)}$$

$$= \limsup_{0<r\to 0}\left(\sup_{x\in B_{|||\cdot|||}(y,r)} \frac{m_d\Big(B \cap B_{|||\cdot|||}(y,r)\Big)}{m_d\Big(B_{|||\cdot|||}(y,r)\Big)}\right)$$

$$\leq \limsup_{0<r\to 0} \left(\left(\frac{c_2}{c_1}\right)^d \cdot \frac{m_d\Big(B \cap B_{\|\cdot\|}(x\,,\frac{2\,r}{c_1})\Big)}{m_d\Big(B_{\|\cdot\|}(x\,,\frac{2\,r}{c_1})\Big)} \right)$$

$$= \left(\frac{c_2}{c_1}\right)^d \cdot D_{\|\cdot\|,B}(x) = 0\,.$$

□

Replacing in the above statement B by $\mathbb{R}^d \setminus B$, we obtain its counterpart for 1:

Let $B \subset \mathbb{R}^d$ be a Borel set, and $x \in \mathbb{R}^d$. If the density $D_{\|\cdot\|,B}(x)$ exists and is equal to 1 for some norm $\|\cdot\|$ on $\mathbb{R}^d$, then, for every norm $|||\cdot|||$ on $\mathbb{R}^d$, the limit

$$\lim_{\substack{0<r\to 0 \\ x \in B_{|||\cdot|||}(y\,,r)}} \frac{m_d\Big(B \cap B_{|||\cdot|||}(y\,,r)\Big)}{m_d\Big(B_{|||\cdot|||}(y\,,r)\Big)}$$

exists and is equal to 1.

In particular, the existence of density 1 or 0 of B at x is equivalent to the existence of density 1 or 0 of B at x in the strong sense (2) and does not depend on the used norm.

The exceptional points

Let us call $x \in \mathbb{R}^d$ *exceptional point for a Borel set* $B \subset \mathbb{R}^d$ *with respect to the norm* $\|\cdot\|$ *on* $\mathbb{R}^d$ if

- either the density $D_{\|\cdot\|,B}(x)$ does not exist (that is $\underline{D}_{\|\cdot\|,B}(x) < \overline{D}_{\|\cdot\|,B}(x)$),

- or $0 < D_{\|\cdot\|,B}(x) < 1$.

Let us call a Borel set $B \subset \mathbb{R}^d$ *non-trivial* if $m_d(B) > 0$ and $m_d(\mathbb{R}^d \setminus B) > 0$, that is if the density function $D_{\|\cdot\|,B}$ is not a well defined constant function (necessarily 1 or 0, does not matter wich norm on $\mathbb{R}^d$ is used).

For a non-trivial Borel set $B \subset \mathbb{R}^d$ exceptional points must exist with respect to any norm on $\mathbb{R}^d$.

In the case $d = 1$ this is particularly easy to see:

Indeed, assuming that no exceptional points exist, the density of B is defined and equal to 1 or 0 at every point $x \in \mathbb{R}$. But then, as we have seen, the density of B exists in the strong sense (2) everywhere, so the function

$$f_a : (a, +\infty) \ni t \longmapsto m_1\Big(B \cap (a,t)\Big)$$

is derivable for each $a \in \mathbb{R}$ and the range of its derivative is contained in $\{0\,,1\}$. Since derivatives have the Darboux property, this is possible only if the derivative of f_a is constant, what is not true for every a.

The above reasoning can be adapted (using, for example, the Darboux type theorem from [5]) also to the case of Borel sets in $\mathbb{R}^d$ with $d > 1$.

According to the Lebesgue density theorem, the set of all exceptional points is narrow, of zero Lebesgue measure. Nevertheless, one can look, in the case of non-trivial Borel sets, for particular exceptional points, for example, for a point at which the density is $1/2$. This happens, for example, in the case of the set $[0,+\infty) \subset \mathbb{R}$.

Exceptional points in $\mathbb{R}$

On $\mathbb{R}$ we consider only the usual norm, the absolute value, and denote, for $B \subset \mathbb{R}$ a Borel set and $x \in \mathbb{R}$, simply

$$\underline{D}_B(x) := \liminf_{0<r\to 0} \frac{m_1\Big(B \cap [x-r, x+r]\Big)}{2r},$$
$$\overline{D}_B(x) := \limsup_{0<r\to 0} \frac{m_1\Big(B \cap [x-r, x+r]\Big)}{2r}.$$

V. I. Kolyada has found in 1983 (see [3]):

There exists a constant $\delta > 0$ such that, for every non-trivial Borel set $B \subset \mathbb{R}$, there exists a point $x \in \mathbb{R}$ such that

$$\delta \le \underline{D}_B(x) \le \overline{D}_B(x) \le 1-\delta. \qquad (3)$$

He found out that the above statement works with $\delta = 1/4$. The optimal value for δ was established much later, due to the successive work of A. Szenes ([6]), M. Csörnyei, J. Grahl, T. O'Neil ([2]) and, finally, O. Kurka ([4]) (we notice that all the above work uses a discretization of the problem, whose idea is attributed by A. Szenes to M. Laczkovich). It is namely the only real root of the polynomial equation

$$8\lambda^3 + 8\lambda^2 + \lambda - 1 = 0,$$

namely $\delta_1 = 0,268486...$ (see [4], Theorem 1.1).

It would be interesting to find out, whether also for $d > 1$ and for any norm $\| \cdot \|$ on $\mathbb{R}^d$, there is a constant $\delta > 0$ (depending of course on d and $\| \cdot \|$), such that, for every non-trivial Borel set $B \subset \mathbb{R}^d$, we have

$$\delta \leq \underline{D}_{\| \cdot \|, B}(x) \leq \overline{D}_{\| \cdot \|, B}(x) \leq 1 - \delta$$

with an appropriate $x \in \mathbb{R}^d$.

Exceptional points in $\mathbb{R}^d$ for arbitary d

Looking on (3), we can ask (for "symmetry reason"?): does it exist, for every non-trivial Borel set $B \subset \mathbb{R}$, some $x \in \mathbb{R}$ satisfying

$$\delta \leq \underline{D}_B(x) \leq \frac{1}{2} \leq \overline{D}_B(x) \leq 1 - \delta\,.$$

Recently, R. Peirone and I have proved the following statement, implying a partial answer to the above question:

Let $B \subset \mathbb{R}^d$ be a Borel set, and $\emptyset \neq U \subset \mathbb{R}^d$ a connected open set such that $m_d(B \cap U) > 0$ and $m_d\Big(U \setminus (B \cap U)\Big) > 0$ (we could say that $B \cap U$ is non-trivial in U). Then there exists some $x \in U$ such that

$$\liminf_{0<r\to 0} \frac{m_d\Big(B \cap (x + r\,F)\Big)}{m_d(x + r\,F)} \leq \frac{1}{2}$$

$$\leq \limsup_{0<r\to 0} \frac{m_d\Big(B \cap (x + r\,F)\Big)}{m_d(x + r\,F)}$$

for every bounded Borel subset F of $\mathbb{R}^d$ with $m_d(F) > 0$, which is symmetric with respect to the origin ; in particular, for this x we have

$$\underline{D}_{\|\cdot\|,B}(x) \leq \frac{1}{2} \leq \overline{D}_{\|\cdot\|,B}(x)$$

for every norm $\|\cdot\|$ on $\mathbb{R}$.

In particular, if $B \subset \mathbb{R}^d$ is a non-trivial Borel set such that, for every $x \in \mathbb{R}^d$, the density of B exists at x with respect to some norm $\|\cdot\|_x$ on $\mathbb{R}^d$ depending on x, then there exists some $x \in \mathbb{R}^d$ such that

$$D_{\|\cdot\|_x,B}(x) = \frac{1}{2}.$$

In the case $\|\cdot\|_x = \|\cdot\|_2$ for all $x \in \mathbb{R}^d$, where $\|\cdot\|_2$ stands for the Euclidean norm on $\mathbb{R}^d$, the above statement was proved by A. Andretta, R. Camerlo, C. Costantini ([1]).

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