First Analysis Mathematica International Conference

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Korevaar-Schoen's directional energy and Ambrosio's regular Lagrangian flows

Budapest, 12 - 17 August 2019

Essence of the matter

Problem. Let $p \in (1, \infty)$ and $v \in \mathbb{R}^d$ be a vector. How can we define the class of "directional Sobolev maps"from $\mathbb{R}^d$ to $\mathbb{R}^d$?

Distributional approach. Given a map $u : \mathbb{R}^d \to \mathbb{R}^d$ and a vector $v \in \mathbb{R}^d$, we say that u belongs to the Sobolev space $W_p^v(\mathbb{R}^d, \mathbb{R}^d)$ if $u \in L_p(\mathbb{R}^d, \mathbb{R}^d)$ and $\exists h \in L_1^{loc}(\mathbb{R}^d, \mathbb{R}^d)$ s.t. $\forall \varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$

$$\int (h, \psi)\, d\mathcal{L}_d = -\int (u, \partial_v \psi)\, d\mathcal{L}_d.$$

We shall call h as distributional derivative of u and denote it by $\partial_v u$.

Metric approach (Korevaar-Schoen). Given $p \in (1, \infty)$ and $v \in \mathbb{R}^d$ we write $u \in \mathrm{KS}_p^v(\mathbb{R}^d, \mathbb{R}^d)$ if $u \in L_p(\mathbb{R}^d, \mathbb{R}^d)$ and

$$\sup_{\varphi \in C_c^\infty(\mathbb{R}^d)} \overline{\lim_{\varepsilon \to 0}} \int \varphi(x) \frac{\|u(x + \varepsilon v) - u(x)\|^p}{\varepsilon^p}\, d\mathcal{L}_d(x) < +\infty.$$

Essence of the matter

It is not difficult to establish the following

Proposition. Let $p \in (1, \infty)$ and $v \in \mathbb{R}^d$. Then $u \in W_p^v(\mathbb{R}^d, \mathbb{R}^d) \Leftrightarrow u \in \mathrm{KS}_p^v(\mathbb{R}^d, \mathbb{R}^d)$. Furthermore,

$$\frac{u(\cdot + \varepsilon v) - u(\cdot)}{\varepsilon} \to \partial_v u, \quad \varepsilon \to 0 \quad \text{in} \quad L_p(\mathbb{R}^d, \mathbb{R}^d).$$

Due to the fact that the second definition is purely metric and intrinsic it is reasonable to pose the following

Problem. Is it possible to introduce Sobolev maps in "the spirit of Korevaar-Schoen"when both the target space and the source space are nonsmooth?

Classical Korevaar-Schoen space

Let $\mathrm{M} = (\mathrm{M}, g)$ be a compact smooth Riemannian manifold, Z be a smooth vector field on M.
We built the flow Fl_t^Z which arises as a solution of the ODE system

$$\begin{cases} \partial_t \mathrm{Fl}_t^Z(x) = Z(\mathrm{Fl}_t^Z(x)), \quad (x,t) \in \mathrm{M} \times (0,T); \\ \mathrm{Fl}_0^Z(x) = x, \quad x \in \mathrm{M}. \end{cases}$$

Fix $p \in (1, \infty)$ and complete metric space $(\mathrm{Y}, \mathrm{d_Y})$. Given a map $u \in L_p(\mathrm{M}, \mathrm{Y})$ we set for every $(x, \varepsilon) \in \mathrm{M} \times (0, T)$

$$e^Z_{p,\varepsilon}[u](x) := \frac{\mathrm{d}^p_{\mathrm{Y}}(u(x), u(\mathrm{Fl}^Z_\varepsilon(x)))}{\varepsilon^p}.$$

For every $\varepsilon > 0$ consider the functional

$$E^Z_{p,\varepsilon}[u](\varphi) := \int \varphi(x) e^Z_{p,\varepsilon}[u](x)\, d\mu_g(x), \quad \varphi \in C_c(\mathrm{M}).$$

Classical Korevaar-Schoen space

The following definition was firstly introduced in [1]

Definition 1 Given $p \in (1, \infty)$, a smooth vector field Z and complete metric space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$, we say that a map u belongs to $\mathrm{KS}^1_p(\mathrm{M}, \mathrm{Y})$ if

(1) $u \in L_p(\mathrm{M}, \mathrm{Y})$;

(2) $E^Z_p[u] := \overline{\lim}_{\varepsilon \to 0} E^Z_{p,\varepsilon}[u] < +\infty$.

In [1] various properties of directional energy functionals $E^Z_p[u]$ were established for the purposes of proving Lipschitz regularity of harmonic maps between Riemannian manifolds and CAT(0) spaces.

In [2] Lipschitz regularity of harmonic maps was established when X has nonnegative curvature in the sense of Alexandrov.

Nicola Gigli initiated a big project aiming at developing a full theory of Sobolev maps between singular spaces. More precisely, we would like to replace Riemannian manifold to a singular space with curvature bounded from below in some generalized sense.

$CD(K,\infty)$ spaces

Let $(\mathsf{X}, \mathsf{d}_\mathsf{X})$ be a complete separable metric space. Let $\mathcal{P}(\mathsf{X})$ be the space of all probability measures and $\mathcal{P}_2(\mathsf{X}) \subset \mathcal{P}(\mathsf{X})$ all measures with finite second moment. For $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$

$$W_2^2(\mu_0,\mu_1) := \inf_\gamma \iint \mathsf{d}_\mathsf{X}^2(x,y) d\gamma(x,y), \tag{1}$$

inf is taken over all $\gamma \in \mathcal{P}(\mathsf{X}\times\mathsf{X})$ s.t. $(\Pi_1)_\sharp\gamma = \mu_0$, $(\Pi_2)_\sharp\gamma = \mu_1$. We set

$$\mathrm{Ent}_\mathfrak{m}(\mu) := \begin{cases} \int \rho \ln \rho \, d\mathfrak{m}, & \mu = \rho\mathfrak{m} \text{ and } (\rho\ln\rho)^- \in L_1(\mathfrak{m}); \\ +\infty & \text{otherwise} \end{cases} \tag{2}$$

Definition 2 (Lott-Sturm-Villany) Given $K \in \mathbb{R}$, we say that a m.m.s $(\mathsf{X}, \mathsf{d}_\mathsf{X}, \mathfrak{m})$ is $CD(K,+\infty)$ space if $\forall \mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X}) \cap D(\mathrm{Ent}(\mathfrak{m}))$ $\exists$ a geodesic $\mu_t : [0,1] \to \mathcal{P}_2(\mathsf{X})$ s.t.

$$\mathrm{Ent}(\mu_t) \leqslant (1-t)\,\mathrm{Ent}(\mu_0) + t\,\mathrm{Ent}(\mu_1) - \frac{K}{2} t(1-t) d_W^2(\mu_0,\mu_1). \tag{3}$$

Given a metric space (X, d_X), we introduce the slope of a function $f : X \to \mathbb{R}$

$$\operatorname{lip} f(x) := \begin{cases} \overline{\lim\limits_{y \to x}} \frac{|f(x)-f(y)|}{d_X(x,y)}, & x \text{ is not an isolated point,} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3 (J. Cheeger) Given a m.m.s $X = (X, d_X, \mathfrak{m})$ and a number $p \in (1, \infty)$, the Cheeger energy $\mathrm{Ch}_p : L_p(X) \to [0, +\infty]$ is defined by

$$\begin{aligned} &\mathrm{Ch}_p(f) := \\ &\inf\{ \underline{\lim}_{n \to \infty} \int (\operatorname{lip} f_n)^p \, d\mathfrak{m} : \{f_n\} \subset \mathrm{LIP}(X) \text{ and } f_n \to f \text{ in } L_p(X)\} \\ &= \int |\nabla f|_p^p \, d\mathfrak{m}. \end{aligned} \tag{4}$$

Sobolev Spaces and $RCD(K,\infty)$ condition

Definition 4 (J. Cheeger) Given a parameter $p \in (1,\infty)$ and a m.m.s X we define the Cheeger-Sobolev space $W_p^1(\mathrm{X})$ as finiteness domain of Ch_p equipped with the norm

$$\|f\|_{W_p^1(\mathrm{X})} := \|f\|_{L_p(\mathrm{X})} + \Big(\mathrm{Ch}_p(f)\Big)^{\frac{1}{p}}. \tag{5}$$

It is easy to show that in the case $\mathrm{X} = \mathbb{R}^n$ with $\mathfrak{m} = \mathcal{L}_n$ the Cheeger-Sobolev spaces coincide with classial Sobolev spaces. Unfortunately, $W_2^1(\mathbb{R}^n, \|\cdot\|_q, \mathcal{L}_n)$ are not Hilbert for $q \neq 2$.

Definition 5 (Ambrosio-Gigli-Savare) Given $K \in \mathbb{R}$, we say that a m.m.s $\mathrm{X} = (\mathrm{X}, \mathrm{d}_\mathrm{X}, \mathfrak{m})$ satisfies the Riemannian Curvature Dimension condition $\mathrm{RCD}(K,\infty)$ if it is a $\mathrm{CD}(K,\infty)$ m.m.s. and the space $W_2^1(\mathrm{X})$ is Hilbert.

In the context of smooth manifolds Definition 5 $\Leftrightarrow$ $\mathrm{Ric} \geqslant K$.

Vector fields on $\mathrm{RCD}(K, \infty)$ spaces

Fix a m.m.s. $\mathrm{X} = (\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m})$.
Definition 6 (Weaver-Gigli) We say that a linear functional $Z : \mathrm{LIP}(\mathrm{X}) \to L^0(\mathrm{X})$ is a vector field (or derivation) if

$$Z(fg) = Z(f)g + fZ(g), \quad \forall f, g \in \mathrm{LIP}(\mathrm{X}),$$

we write $Z \in L_p(T\mathrm{X})$, $p \in [1, \infty]$ if $\exists g \in L_p(\mathrm{X})$ s.t. $\forall f \in \mathrm{LIP}(\mathrm{X})$

$$Z(f) \leqslant g|\nabla f|_p, \quad \mathfrak{m} - a.e.$$

Given $p \in [1, \infty]$, we say that a family $\{Z_t\}_{t\in[0,1]} \subset L_p(T\mathrm{X})$ is a time dependent vector field if $\forall f \in W^1_{p'}(\mathrm{X})$ the map $(t, x) \to Z_t(f)(x)$ is measurable with respect to $\mathcal{B}([0, 1] \times \mathrm{X})$.

Regular Lagrangian Flows

Definition 7 (Ambrosio)Let $Z = Z_t$ be a time dependent vector field. A Regular Lagrangian Flow of Z is a Borel map $\mathrm{Fl}^Z : [0,1] \times \mathrm{X} \to \mathrm{X}$ if

(1) $\mathrm{Fl}^Z_0(x) = x$ and $\mathrm{Fl}^Z_\cdot(x) \in C([0,1], \mathrm{X})$ for every $x \in \mathrm{X}$;

(2) there exists $C > 0$ – compressibility constant, s.t.

$$\left(\mathrm{Fl}^Z_t\right)_\sharp \mathfrak{m} \leqslant C\mathfrak{m};$$

(3) for every $f \in W^1_2(\mathrm{X})$ it holds: for $\mathfrak{m}$-a.e. x the map $t \to f(\mathrm{Fl}^Z_t(x))$ is in $W^1_1((0,1))$ with

$$\partial_t f(\mathrm{Fl}^Z_t(x)) = Z_t(f) \circ \mathrm{Fl}^Z_t(x) \quad \text{a.e. } t \in [0,1].$$

Theorem (Ambrosio and Trevisan) Let X be an $\mathrm{RCD}(K,\infty)$ space. Then, for a sufficiently nice vector field Z called regular vector field there exists a unique Regular Lagrangian Flow Fl^Z.

Korevaar-Schoen again

Let $\mathsf{X} = (\mathsf{X}, \mathsf{d}_\mathsf{X}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ space and $\mathsf{Y} = (\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ be a complete metric space. Let Z be a regular autonomous vector field on X and Fl^Z be the unique R.L.F. associated with Z.

Definition 8 (Gigli-T. 2019 [3]) Given $p \in (1, \infty)$ we say that $u \in \mathsf{KS}_p^Z(\mathsf{X}, \mathsf{Y})$ iff $u \in L_p(\mathsf{X}, \mathsf{Y})$

$$\sup_{\varphi} E_p^Z[u](\varphi) = \sup_{\varphi} \overline{\lim_{\varepsilon \to 0}} \int \varphi(x) \frac{\mathsf{d}_\mathsf{Y}^p(u(x), u(\mathsf{Fl}_\varepsilon^Z(x)))}{\varepsilon^p} \, d\mathfrak{m}(x) < \infty.$$

Density of the energy

Theorem 1(Gigli-T. [3]) Let $\mathsf{X} = (\mathsf{X}, \mathsf{d}_\mathsf{X}, \mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space and $\mathsf{Y} = (\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ be a complete metric space. Let Z be a regular autonomous vector field on X and Fl^Z be the unique R.L.F. associated with Z. Given $p \in (1,\infty)$, the following are equivalent:
(1) It holds $u \in \mathsf{KS}_p^Z(\mathsf{X},\mathsf{Y})$;
(2) $\exists G \in L_p(\mathsf{X})$ s.t. $\forall 0 \leqslant t \leqslant s \leqslant 1$

$$\mathsf{d}_\mathsf{Y}(u \circ \mathsf{Fl}^Z_t, u \circ \mathsf{Fl}^Z_s) \leqslant \int_t^s G \circ \mathsf{Fl}^Z_r \, dr, \quad \mathfrak{m} - a.e..$$

This means that the curve $t \to u \circ \mathsf{Fl}^Z_t \in AC^p_{loc}([0,1], L_p(\mathsf{X},\mathsf{Y}))$. Moreover if these hold $\exists e^Z_p[u] \in L_p(\mathsf{X})$ called the energy density function s.t.

$$\frac{\mathsf{d}_\mathsf{Y}(u(\cdot), u \circ \mathsf{Fl}^Z_\varepsilon(\cdot))}{\varepsilon} \to e^Z_p[u](\cdot), \quad \varepsilon \to 0 \quad \text{in} \quad L_p(\mathsf{X}).$$

Triangle inequality for the energy

Theorem 2(Gigli-T. [3]) Let $\mathsf{X} = (\mathsf{X}, \mathsf{d}_\mathsf{X}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ space for some $K \in \mathbb{R}$ and $\mathsf{Y} = (\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ be a complete metric space. Given $p \in (1, \infty)$, the following is true:
(1) Let Z be a regular vector field and $u \in \mathsf{KS}_p^Z(\mathsf{X}, \mathsf{Y})$. Then $u \in \mathsf{KS}_p^{\alpha Z}(\mathsf{X}, \mathsf{Y})$ for each $\alpha \in \mathbb{R}$ and

$$e_p^{\alpha Z}[u] = |\alpha| e_p^Z[u]. \tag{6}$$

(2) For every regular autonomous vector fields Z_1, Z_2 and $u \in \mathsf{KS}_p^{Z_1}(\mathsf{X}, \mathsf{Y}) \cap \mathsf{KS}_p^{Z_2}(\mathsf{X}, \mathsf{Y})$ it holds that $u \in \mathsf{KS}_p^{Z_1 + Z_2}(\mathsf{X}, \mathsf{Y})$ and

$$e_p^{Z_1+Z_2}[u] \leqslant e_p^{Z_1}[u] + e_p^{Z_2}[u], \quad \mathfrak{m} - a.e.. \tag{7}$$

CAT(0) spaces

Definition 9 A complete metric space $(\mathrm{Y}, \mathrm{d_Y})$ is said to be CAT(0) space if:

(1) $(\mathrm{Y}, \mathrm{d_Y})$ is a length space, that is, $\forall P, Q \in Y$, the distance $\mathrm{d_Y}(P, Q)$ is realized as the length of rectifiable curve connecting P and Q;
(2) $\forall P, Q, R \in \mathrm{Y}$ and geodesics $\gamma_{P,Q}$, $\gamma_{Q,R}$, $\gamma_{R,P}$ with lengthes r, p, q respectively, the following comparison property is to hold: For any $\lambda \in (0, 1)$ write Q_λ for the point on γ_{QR} which a fraction λ of the distance from Q to R. That is,

$$\mathrm{d_Y}(Q_\lambda, Q) = \lambda p, \quad \mathrm{d_Y}(Q_\lambda, R) = (1 - \lambda)p.$$

Then the metric distance $\mathrm{d_Y}(P, Q_\lambda)$ is bounded above by the Euclidean distance $|\overline{P} - \overline{Q}_\lambda|$, i.e.

$$\mathrm{d}^2_{\mathrm{Y}}(P, Q_\lambda) \leqslant (1 - \lambda)\,\mathrm{d}^2_{\mathrm{Y}}(P, Q) + \lambda\,\mathrm{d}^2_{\mathrm{Y}}(P, R) - \lambda(1 - \lambda)\,\mathrm{d}^2_{\mathrm{Y}}(Q, R).$$

Parallelogram identity for the directional energy

Theorem 3(Gigli-T. [3]) Let $K \in \mathbb{R}$, $(\mathsf{X}, \mathsf{d}_\mathsf{X}, \mathfrak{m})$ be $\mathsf{RCD}(K, \infty)$ space, Z_1, Z_2 two regular vector fields on it. Let $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ be a $\mathsf{CAT}(0)$ space and $u \in \mathsf{KS}_p^{Z_1}(\mathsf{X}, \mathsf{Y}) \cap \mathsf{KS}_p^{Z_2}(\mathsf{X}, \mathsf{Y})$. Then

$$
\begin{aligned}
&|e_2^{(Z_1+Z_2)}[u]|^2 + |e_2^{(Z_1-Z_2)}[u])|^2 = \\
&2\left(|e_2^{Z_1}[u]|^2 + |e_2^{Z_2}[u]|^2\right) \qquad \mathfrak{m} - a.e..
\end{aligned}
\tag{8}
$$

Thank you for your attention

THANK YOU FOR YOUR ATTENTION !

(1) Korevaar, Nicholas J.; Schoen, Richard M., *Sobolev spaces and harmonic maps for metric space targets*. Comm. Anal. Geom. 1 (1993), no. 3–4, 561–659.

(2) H.-C. Zhang and X.-P. Zhu, *Lipschitz continuity of harmonic maps between Alexandrov spaces*, Invent. Math., 211 (2018), pp. 863–934.

(3) Gigli Nicola and Tyuenev Alexander, *Korevaar-Schoen's directional energy and Ambrosio's Regular Lagrangian Flows*. https://arxiv.org/abs/1901.03564.