ON PECHERSKIJ-RÉVÉSZ'S THEOREM

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The talk is motivated mainly by
Szilard Gy. Révész,
Rearrangement of Fourier series and Fourier series whose terms have random signs.
Acta Math. Hung. 63:4 (1994), 395–402.
The talk is based mainly on
S.A. Chobanyan, G. J. Giorgobiani, V.I. Tarieladze,
Signs and Permutations:Two Problems of the Function Theory.
Proceedings of A. Razmadze Mathematical Institute Vol. 160(2012), 25–34.

In what follows Y will stand for a real or complex Banach space and $C([-\pi,\pi],Y)$ for the set of all continuous functions $f : [-\pi,\pi] \to Y$ such that $f(-\pi) = f(\pi)$. This set with respect to the point-wise operations and norm

$$f \mapsto \|f\| := \sup_{t\in[-\pi,\pi]} \|f(t)\|_Y$$

is a Banach space.

For $f \in C([-\pi,\pi],Y)$ we write:

$$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt,\ b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt,\ n = 0,1,\dots,$$

and define the functions $A_n(f) : [-\pi,\pi] \to Y$, $n = 0,1,\dots$ by the equalities:

$$A_0(f)(t) = \frac{1}{2}a_0(f),\ A_n(f)(t) = a_n(f)\cos nt + b_n(f)\sin nt \quad \forall n \in \mathbb{N}, \forall t \in [-\pi,\pi]$$

and call

$$A_0(f) + \sum_{n=1}^{\infty} A_n(f)$$

the (trigonometric) Fourier series of f.

Let us denote

- by $\mathbf{P}([-\pi,\pi],Y)$ the set of all $f \in C([-\pi,\pi],Y)$ for which the corresponding Fourier series converges point-wise to f;
- by $\mathbf{P}_{\mathrm{men}}([-\pi,\pi],Y)$ the set of all $f \in C([-\pi,\pi],Y)$ for which **some subsequence** of the sequence

$$\left(\sum_{k=0}^{n} A_k(f)\right)_{n\in\mathbb{N}}$$

  converges point-wise to f;
- by $\mathbf{U}([-\pi,\pi],Y)$ the set of all $f \in C([-\pi,\pi],Y)$ for which the corresponding Fourier series converges uniformly to f;
- by $\mathbb{U}_{\mathrm{ul}}([-\pi,\pi],Y)$ the set of all $f \in C([-\pi,\pi],Y)$ for which there is a permutation $\sigma:\mathbb{N}\to\mathbb{N}$ such that the series $A_0(f)+\sum_{n=1}^{\infty} A_{\sigma(n)}(f)$ converges uniformly to f.

We know:

- $\mathbf{P}([-\pi,\pi],\mathbb{R}) \neq C([-\pi,\pi],\mathbb{R})$ (Du Bois Reymond, Lebesgue, Fejer).
- $\mathbf{P}([-\pi,\pi],\mathbb{R}) \neq \mathbf{U}([-\pi,\pi],\mathbb{R})$ (H. Lebesgue, 1905) and hence, $\mathbf{U}([-\pi,\pi],\mathbb{R}) \neq C([-\pi,\pi],\mathbb{R})$ as well.
- $\mathbf{P}_{\mathrm{men}}([-\pi,\pi],\mathbb{R}) \neq C([-\pi,\pi],\mathbb{R})$ (according to [6] this was shown by D. Menshov in 1944).

In view of the last result, it is a pleasant surprise that permutations may help!

Theorem 1 (Szilard Gy. Révész; cf. [6, Theorem 1]) *For any $f \in C([-\pi,\pi],\mathbb{R})$ there is a permutation $\sigma:\mathbb{N}\to\mathbb{N}$ such that* **some subsequence** *of the sequence*

$$\left(A_0(f)+\sum_{k=1}^{n} A_{\sigma(k)}(f)\right)_{n\in\mathbb{N}}$$

converges uniformly to f.

For some known (classical and non-classical yet) conditions for uniform convergence see [1].

Ulyanov [9] and Révész [6] conjectured that

$$\mathbb{U}_{\mathrm{ul}}([-\pi,\pi],\mathbb{R}) = C([-\pi,\pi],\mathbb{R})\,.$$

According to
S.B. Stechkin,
On a problem by P.L. Ulyanov (in Russian),
Uspehi Mat. Nauk, 1962, **17**, 5(107), 143–144,
P.L. Ulyanov this conjecture has formulated already in 1962.

This conjecture remains open so far. There are several results dealing with the Ulyanov conjecture. In [8] and in [6] one can find an example of a function $f \in C([-\pi,\pi],\mathbb{R}) \setminus \mathbf{P}([-\pi,\pi],\mathbb{R})$, such that $f \in \mathbb{U}_{\mathrm{ul}}([-\pi,\pi],\mathbb{R})$. Konyagin [3] has proved that if the modulus of continuity of a function $f \in C([-\pi,\pi],\mathbb{R})$ satisfies a weakened Dini-Lipschitz type condition, then $f \in \mathbb{U}_{\mathrm{ul}}([-\pi,\pi],\mathbb{R})$.

Definition 2. We say that a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of a normed space X *satisfies the Rademacher condition*, if the series $\sum_{k=1}^{\infty} x_k r_k(\omega)$ converges in X for $\mathbb{P}$-almost every $\omega \in \Omega$.

Theorem 3 (*Pecherskij-Révész's theorem*; cf. [7, Theorem 1] and [5, Theorem 2]) *Let $f \in C([-\pi,\pi],\mathbb{R})$ be such that the sequence $(A_k(f))_{k\in\mathbb{N}}$ satisfies the Rademacher condition in $C([-\pi,\pi],\mathbb{R})$. Then $f \in \mathbb{U}_{\mathrm{ul}}([-\pi,\pi],\mathbb{R})$.*

To state the main result of this talk we need one more definition.

Definition 4. We say that a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of a normed space X *satisfies the* (σ,θ)*-condition*, if for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ there exists a collection of signs $\theta = (\theta_1, \theta_2, \dots)$ such that the series $\sum_{i=1}^{\infty} x_{\sigma(i)}\theta_i$ converges in X.

Since the sequences satisfying the Rademacher condition satisfy the (σ,θ)-condition, the following result formally is a refinement of Theorem 3 even for $Y = \mathbb{R}$.

Theorem 5. *Let Y be a Banach space and $f \in C([-\pi,\pi],Y)$ be such that the sequence $(A_k(f))_{k\in\mathbb{N}}$ satisfies the (σ,θ)- condition in $C([-\pi,\pi],Y)$. Then $f \in \mathbb{U}_{\mathrm{ul}}([-\pi,\pi],Y)$.*

Proof. Fix a function $f \in C([-\pi,\pi],Y)$ such that

(1) *the sequence $(A_k(f))_{k\in\mathbb{N}}$ satisfies the (σ,θ)- condition in $C([-\pi,\pi],Y)$.*

Write $S_n(f) = \sum_{i=0}^{n} A_i(f)$, $n = 0, 1, \dots$. By the Fejer theorem,

(2) *the sequence*

$$\frac{1}{n+1}\sum_{k=0}^{n} S_k(f),\ n = 1, 2, \dots$$

converges in $C([-\pi,\pi],Y)$ to f.

From (1) and (2) according to [5, Corollary 4] it follows that there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that the series $A_0(f) + \sum_{n=1}^{\infty} A_{\sigma(n)}(f)$ converges in $C([-\pi,\pi],Y)$ to f. Consequently, $f \in \mathbb{U}_{\mathrm{ul}}([-\pi,\pi],Y)$. $\square$

Remark 6. Let

- $C_{\mathrm{rad}}([-\pi,\pi],Y)$ be the set of all $f \in C([-\pi,\pi],Y)$ such that the sequence $(A_k(f))_{k\in\mathbb{N}}$ satisfies the Rademacher condition in $C([-\pi,\pi],Y)$,
- $C_{\sigma,\theta}([-\pi,\pi],Y)$ be the set of all $f \in C([-\pi,\pi],Y)$ such that the sequence $(A_k(f))_{k\in\mathbb{N}}$ satisfies the (σ,θ)- condition in $C([-\pi,\pi],Y)$.

Then

(a) Theorem 3 asserts that $C_{\mathrm{rad}}([-\pi,\pi],\mathbb{R}) \subset \mathbb{U}_{\mathrm{ul}}([-\pi,\pi],\mathbb{R})$, while [7, Theorem 2] tells us that this inclusion is strict.

(b) By Theorem 5 for any Banach space Y we have that

$$C_{\mathrm{rad}}([-\pi,\pi],Y) \subset C_{\sigma,\theta}([-\pi,\pi],Y)$$

and in [2] we conjectured that $C_{\sigma,\theta}([-\pi,\pi],\mathbb{R}) = C([-\pi,\pi],\mathbb{R})$.

(c) If the conjecture from (b) is true, then Theorem 5 would imply the positive answer to Ulyanov's conjecture. Recently we were able to show that even

$$\mathbb{U}([-\pi,\pi],\mathbb{R}) \setminus C_{\sigma,\theta}([-\pi,\pi],\mathbb{R}) \neq \emptyset\,.$$

It follows that the conjecture from (b) is false and, consequently, the Ulyanov's conjecture remains open.

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