On the Voice transform induced by a representation of the dyadic and 2-adic Blaschke group

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Content

Motivation for dyadic and 2-adic analysis

- Why are non-Archimedean local fields important?
- Problems with the practical applications of the classical fields $\mathbb{R}$ and $\mathbb{C}$: in science there are absolute limitations on measurements like Plank time, Plank length, Plank mass.

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  The use of real time and space-time coordinates in mathematical physics leads to some problems with the Archimedean axiom on the microscopic level.
- The Archimedian axiom... we can measure arbitrary small distances.
- But a measurement of distances smaller than the Planck length is impossible.

Motivation for dyadic and 2-adic analysis

- VLADIMIROV, V.S., VOLOVICH, I.V. AND ZELNOV, E.I., *p-adic Analysis and Mathematical Physics*, Series in Soviet and East European Mathematics, Vol. 10, World Scientific Publishers, Singapore-New Jersey -London -Hong Kong, (1994).
  Volovich: some non-Archimedean normed fields have to be used for a global space-time theory in order to unify both microscopic and macroscopic physics. Let us base physics on a coalition of non-Archimedean normed fields and classical fields as $\mathbb{R}$ or $\mathbb{C}$.
- As $p \to \infty$, many of the fundamental functions of p-adic analysis approach their counterparts in classical analysis.

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  - two different balls are either disjoint or the one is contained in the other one (splitting property).
  - the collection of dyadic intervals have a hierarchical structure: every interval consists of two disjoint interval of smaller radius/higher rank (tree property).
- Thus these groups (dyadic and 2-adic) are homeomorphic to a Cantor set on $\mathbb{R}$.
  Volovich: the fractal-like structure of these groups enable their application not only for the description of geometry at small distances, but also for describing chaotic behavior of chaotic systems.

Historical background for Blaschke functions and the Voice transformation

- **The Blaschke functions on $\mathbb{C}$:**
- the open unit disc and its boundary

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}; \ \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\},$$

- the closure of $\mathbb{D}$: $\bar{\mathbb{D}} := \mathbb{D} \cup \mathbb{T}$.
- the disc algebra $\mathfrak{a} := \{F : \bar{\mathbb{D}} \to \mathbb{C} :$ F is analytic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}\}$.
- The Blaschke-function on $\mathbb{C}$ associated to a complex parameter $a \in \mathbb{D}$ is defined by

$$B_a(z) := e^{i\gamma} \frac{z - a}{1 - \bar{a}z} \quad (z \in \mathbb{C}), \tag{1}$$

where $\gamma \in \mathbb{R}$ and $\bar{a}$ is the complex conjugate of $a \in \mathbb{D}$.

Background - Blaschke functions

- $B_a \in \mathfrak{a}$ and B_a is a one-one map from $\mathbb{D}$ onto $\mathbb{D}$, and from $\mathbb{T}$ onto $\mathbb{T}$ for every $a \in \mathbb{D}$.
- The inverse of B_a is also a Blaschke-function:

$$B_a^{-1}(z) = e^{-i\gamma} \frac{z + e^{i\gamma} a}{1 + e^{-i\gamma} \bar{a} z} \quad (z \in \mathbb{C}).$$

Background - Blaschke functions

- B_a can be written in the form

$$B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R}, a \in \mathbb{D}) \tag{2}$$

with the bijection $\beta_a : [-\pi, \pi] \to [-\pi, \pi]$,

$$\beta_a(t) := \gamma + \varphi + 2 \arctan\left(s \tan\left(\frac{t-\varphi}{2}\right)\right),$$

where $a = re^{i\varphi} \in \mathbb{C}$, and $s = \eta(r)$ is defined by means of the bijection $\eta : [0, \infty) \to [0, \infty)$:

$$\eta(r) := \begin{cases} \dfrac{1+r}{1-r} & \text{for } 0 \leq r < 1 \\ \dfrac{r-1}{r+1} & \text{for } 1 \leq r < \infty. \end{cases}$$

- Furthermore, the composition of two Blaschke functions, B_{a_1} and B_{a_2} is a Blaschke function.

Notion of classical Voice transformation

- The common generalization of the Fourier-, wavelet-, Gábor- transforms is the so-called voice-transformation.
- $(G, \cdot)$: a locally compact topological group.
- m: Haar measure.
- A Hilbert-space: $(H; \langle\cdot,\cdot\rangle)$.
- Let $\mathcal{U}$ be **the set of unitary bijections** $U : H \to H$, that is, $\mathcal{U}$ is formed by bounded linear operators U which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$.
- $(\mathcal{U}, \circ)$ is a group, (here $\circ$ denotes the composition operator).

Notion of Unitary representation

- The **unitary representation** of $(G, \cdot)$ on H is defined to be a homomorphism of the group $(G, \cdot)$ on the group $(\mathcal{U}, \circ)$ satisfying

$$U_{xy} = U_x \circ U_y \quad (x, y \in G)$$

$$x \to U_x f \in H \text{ is continuous for all } f \in H \ (x \in G).$$

- The **voice transform** of $f \in H$ generated by the representation U and by the parameter $\rho \in H$ is the (complex- valued) function on G defined by

$$(Vf)(x) := \langle f, U_x \rho \rangle \quad (x \in G, f, \rho \in H).$$

Notion of voice transform

- For any representations $U : G \to \mathcal{U}$ and for each $f, \rho \in H$ the voice transform Vf is a continuous and bounded function on G and $V : H \to C(G)$ is a bounded linear operator.
- Consider the norm $\|F\| := \sup\{|F(x)| : x \in G\}$ on the group G, then the set of continuous bounded functions defined on G form a Banach space.
- From the unitarity of $U_x : H \to H$ follows that, for all $x \in G$ $|(V_\rho f)(x)| = \langle f, U_x \rho \rangle = \|f\| \cdot \|U_x\| = \|f\| \cdot \|\rho\|$; thus $\|V_\rho\| \leq \|\rho\|$.

Notion of multiplier representations

- $F_a : G \to C^* := C \setminus 0 \ (a \in G)$ is called the collection of **multiplier functions** if
$$F_e = 1; F_{a_1 a_2}(x) = F_{a_1}(a_2 \cdot x) F_{a_2}(x) \quad (a_1, a_2, x \in G).$$
- $$(U_a f)(x) := F_{a^{-1}}(x) f(a^{-1}x) \quad (a, x \in G)$$
satisfies
$$U_{a_1} \circ U_{a_2} = U_{a_1 \cdot a_2} \quad (a_1, a_2 \in G),$$
so the above defined $U_a f$ **is a representation of** G on the space of all complex valued functions on G.
- If F_a is continuous and bounded for every $a \in G$, then $L^2_m(G)$ is an invariant subspace and $(U_a)_{a \in G}$ is a representation on $L^2_m(G)$. The representations obtained this way are called **multiplier representations**.

Results of Pap and Schipp, 2006

- If $(U_x)_{x\in G}$ is a unitary multiplier representation of G on $L^2_m(G)$ generated by $F_a \in L^\infty_m(G) \cap C(G)$ $(a \in G)$, then
  $V_\rho \circ U_a = L_a \circ V_\rho$;
  $V_\rho \circ L_a = L_a \circ V_\rho \circ M_a$;
  $(V_\rho f)(x) = \overline{(V_\rho f)(x^{-1})}$ $(a, x \in G, \rho \in H)$.
- The representation of the Blaschke group on $L^2(\mathbb{T})$ generated by the multiplier

$$F_a(e^{it}) := \sqrt{\beta'(t)} \cdot e^{i\frac{\beta_a(t)-t}{2}} \quad (a \in B; t \in \mathbb{I}) \tag{3}$$

is a unitary representation of the Blaschke group on $L^2(\mathbb{T})$.

Notions: dyadic intervals, dyadic squares, Haar measure, L^p- norm

- The set of bits $\mathbb{A} := \{0, 1\}$
- The set of bytes

$$\mathbb{B} := \{a = (a_j, j \in \mathbb{Z}) \mid a_j \in \{0,1\} \text{ and } \lim_{j \to -\infty} a_j = 0\}$$

- The order of a byte $x \in \mathbb{B}$: For $x \neq \theta := (0, 0, \cdots)$ let $\pi(x) = n$ if and only if $x_n = 1$ and $x_j = 0$ for all $j < n$. Set $\pi(\theta) := +\infty$.
- The norm of a byte x is defined by $\|x\| := 2^{-\pi(x)}$ for $x \in \mathbb{B} \setminus \{\theta\}$, and $\|\theta\| := 0$.

Sets

- An interval in $\mathbb{B}$ of rank $n \in \mathbb{Z}$ and center $a \in \mathbb{B}$: $I_n(a) = \{x \in \mathbb{B} : x_j = a_j \text{ for } j < n\}$.
- $\mathbb{I}_n := I_n(\theta) = \{x \in \mathbb{B} : \|x\| \leqq 2^{-n}\}$ for any $n \in \mathbb{Z}$.
- The *unit ball* is $\mathbb{I} := \mathbb{I}_0$.
- $\mathbb{S} := \{x \in \mathbb{B} : \|x\| = 1\} = \{x \in \mathbb{B} : \pi(x) = 0\} = \{x \in \mathbb{I} : x_0 = 1\}$ is the unit sphere.
- Note, that $\circ$ when used between functions, denotes function composition.
- μ: the right and left invariant Haar measure with $\mu(G_m) = 1$.

Characters

- *Rademacher system*: $(r_n, n \in \mathbb{N})$ with $r_n(x) := (-1)^{x_n}$ $(x \in \mathbb{I})$,
- the *Walsh-Paley functions*:

$$w_k(x) = \prod_{n=0}^{\infty} (r_n(x))^{k_n} = (-1)^{\sum_{j=0}^{+\infty} k_j x_j} \; (x \in \mathbb{I}),$$

with dyadic expansion $k = \sum_{j=0}^{\infty} k_j 2^j \in \mathbb{N}$ $(k_j \in \mathbb{A})$.

- $$v_{2^n}(x) := \exp\left(2\pi\imath\left(\frac{x_n}{2} + \frac{x_{n-1}}{2^2} + \cdots + \frac{x_0}{2^{n+1}}\right)\right) \; (x \in \mathbb{I}, n \in$$
- $$v_m(x) = \prod_{j=0}^{\infty} (v_{2^j}(x))^{m_j} \; (m \in \mathbb{N}).$$

.

- $(w_n, n \in \mathbb{N})$ is the character system of $(\mathbb{I}, \overset{\circ}{+})$
- $(v_n, n \in \mathbb{N})$ is the character system of $(\mathbb{I}, \overset{\bullet}{+})$.

The dyadic sum $\overset{\circ}{+}$ and 2-adic sum $\overset{\bullet}{+}$

- The *dyadic sum* $a \overset{\circ}{+} b$ of elements $a, b \in \mathbb{B}$ is defined by

$$a \overset{\circ}{+} b := (a_n + b_n \pmod 2),\ \ n \in \mathbb{Z})\,.$$

- The *2-adic sum* $a \overset{\bullet}{+} b$ of elements $a = (a_n, n \in \mathbb{Z}), b = (b_n, n \in \mathbb{Z}) \in \mathbb{B}$ is defined by $a \overset{\bullet}{+} b := (s_n, n \in \mathbb{Z})$ where the bits $q_n, s_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are obtained recursively as follows:

$$q_n = s_n = 0 \quad \text{for } n < m := \min\{\pi(a), \pi(b)\},$$
$$\text{and}\ \ a_n + b_n + q_{n-1} = 2q_n + s_n \quad \text{for } n \geq m.$$

The dyadic product $\circ$ and 2-adic product $\bullet$

- *dyadic product* $a \circ b$ of elements $a, b \in \mathbb{B}$ is defined by $a \circ b := (c_n, n \in \mathbb{Z}),$ where $c_n = \sum_{k \in \mathbb{Z}} a_k b_{n-k} \pmod 2$ $(n \in \mathbb{Z})$.
- The *2-adic product* of $a, b \in \mathbb{B}$ is $a \bullet b := (p_n, n \in \mathbb{Z})$, where the sequences $q_n \in \mathbb{N}$ and $p_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are defined recursively by

$$q_n = p_n = 0 \quad (n < m := \pi(a) + \pi(b))$$

$$\text{and} \quad \sum_{j=-\infty}^{\infty} a_j b_{n-j} + q_{n-1} = 2q_n + p_n \quad (n \geq m).$$

The additive inverse of $\dot{+}$

The *reflection* x^- of a byte $x = (x_j, j \in \mathbb{Z})$ is defined by:

$$(x^-)_j := \begin{cases} x_j, & \text{for } j \leqq \pi(x) \\ 1 - x_j, & \text{for } j > \pi(x). \end{cases}$$

$e := (\delta_{n0}, n \in \mathbb{Z})$, where δ_{nk} is the Kronecker-symbol.

We will use the following notation: $a \dot{-} b := a \dot{+} b^-$.

The dyadic Blaschke function, 2006

- For $a \in \mathbb{I}_1$ *the dyadic Blaschke function* on $(\mathbb{I}, \overset{\circ}{+}, \circ)$ is defined by:

$$B_a(x) := (x \overset{\circ}{+} a) \circ (e \overset{\circ}{+} a \circ x)^{-1} = \frac{x \overset{\circ}{+} a}{e \overset{\circ}{+} a \circ x} \qquad (x \in \mathbb{I}).$$

- Since $\pi(a) \geqq 1$ and $\pi(x) \geqq 0$, we have $\pi(a \circ x) \geqq 1$, therefore we have $\pi(e \overset{\circ}{+} a \circ x) = 0$, hence $e \overset{\circ}{+} a \circ x \neq \theta$. Thus the function B_a is well-defined on $\mathbb{I}$.
- Note, that $\pi(u \circ v^{-1}) = \pi(u) - \pi(v)$ $(u, v \in \mathbb{B})$, thus

$$\|B_a(x)\| \leqq 1 \text{ if } \|x\| \leqq 1, \text{ and } \|B_a(x)\| = 1 \text{ if } \|x\| = 1. \tag{4}$$

B_a is a bijection on the "disc" and the "torus"

- B_a is a bijection on the unit ball $\mathbb{I}$ and on the unit sphere $\mathbb{S} := \mathbb{S}_0 = \{x \in \mathbb{B} \mid \|x\| = 1\}$.
- $B_a^{-1} = B_a$.
- For $a, b \in \mathbb{I}_1$,

$$B_a(B_b(x)) = B_c(x) \ \ (x \in \mathbb{I}), \text{ where } c = \frac{a \overset{\circ}{+} b}{e \overset{\circ}{+} a \circ b} = B_a(b) \in \mathbb{I}_1 .$$

- The maps B_a $(a \in \mathbb{I}_1)$ form a commutative group with respect to the composition of functions.

The recursive form of the byte $B_a(x)$:

With $y = B_a(x)$ we have $y = x \overset{\circ}{+} a \overset{\circ}{+} y \circ a \circ x$. So,

$$\begin{cases} y_n = 0, \text{ for } n < 0, \\ y_n = x_n + a_n + (y \circ a \circ x)_n \pmod 2, \text{ for } n \geqq 0. \end{cases}$$

As the n-th digit of $y \circ a \circ x$ depends only on a and x_k-s with $k < n$, we have that $y = B_a(x)$ can be written in the form

$$y_n = x_n + a_n + f_n(x_0, \cdots, x_{n-1}) \pmod 2 \tag{5}$$

where the functions $f_n : \mathbb{A}^n \to \mathbb{A}$ $(n = 1, 2, \cdots)$ depend only on the bits of a.

The 2-adic Blaschke-function, 2006

- For $a \in \mathbb{I}_1$ *the 2-adic Blaschke function* on $(\mathbb{I}, \dot{+}, \bullet)$ is defined by:

$$B_a(x) := (x \dot{-} a) \bullet (e \dot{-} a \bullet x)^{-1} = \frac{x \dot{-} a}{e \dot{-} a \bullet x} \qquad (x \in \mathbb{I}).$$

- For $x \in \mathbb{I}$ and $a \in \mathbb{I}_1$ we have that $e \dot{-} a \bullet x \neq \theta$, thus $e \dot{-} a \bullet x$ has a multiplicative inverse in $\mathbb{B}$, and so the function is well-defined.
- $B_a : \mathbb{I} \to \mathbb{I}$ is a bijection for any $a \in \mathbb{I}_1$ on $\mathbb{I}$, and on $\mathbb{S} \subset \mathbb{I}$ as well.
- the inverse of B_a is $B_a^{-1} = B_{a^-}$.
- $\|B_a(x)\| = 1$ iff $\|x\| = 1$.

The Blaschke group

$B_a \circ B_b = B_c$, where $c = \dfrac{a \overset{\bullet}{+} b}{e \overset{\bullet}{+} a \bullet b} \in \mathbb{I}_1 \;\; (a, b \in \mathbb{I}_1)$.

With notation $a \triangleleft b := \dfrac{a \overset{\bullet}{+} b}{e \overset{\bullet}{+} a \bullet b} \in \mathbb{I}_1 \;\; (a, b \in \mathbb{I}_1)$ we have $B_a \circ B_b = B_{a \triangleleft b} \; (a, b \in \mathbb{I}_1)$.

The maps $B_a \; (a \in \mathbb{I}_1)$ form a commutative group with respect to the composition of functions. The identity element: $B_\theta = \imath$, the inverse element of B_a is B_{a^-}.

Definition

Consider $\mathcal{B} := \{B_a, a \in \mathbb{I}_1\}$. $(\mathcal{B}, \circ)$ when using the 2-adic operations is called *the 2-adic Blaschke-group* of $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$. $(\mathcal{B}, \circ)$ when using the dyadic operations is called *the dyadic Blaschke-group* of $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

The recursive form of the byte $B_a(x)$

The digit $y_n = (B_a(x))_n$ can be written in the form

$$y_n = x_n + f_n(x_0, \cdots, x_{n-1}) \pmod 2$$

where the functions $f_n : \mathbb{A}^n \to \mathbb{A}$ $(n = 1, 2, \cdots)$ depend only on the bits of a.

$B_a : \mathbb{I} \to \mathbb{I}$ is measure preserving

The variable transformation $B_a : \mathbb{I} \to \mathbb{I}$ is measure preserving for each $a \in \mathbb{I}_1$. Hence,

$$\int_{\mathbb{I}} f \circ B_a d\mu = \int_{\mathbb{I}} f d\mu \qquad (f \in L^1(\mathbb{I})). \tag{6}$$

An example

For example, the dyadic and 2-adic sums of a and b,

$$a = (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{1}, \overset{2}{0}, \overset{3}{1}, \overset{4}{0}, \overset{5}{1}, \cdots)$$

$$b = (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{0}, \overset{2}{1}, \overset{3}{1}, \overset{4}{1}, \overset{5}{1}, \cdots)$$

are the following: $a \overset{\circ}{+} b = (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{1}, \overset{2}{1}, \overset{3}{0}, \overset{4}{1}, \overset{5}{0}, \cdots)$,

$$a+b = (\cdots, \overset{-1}{0}, \overset{0}{0}, \overset{1}{1}, \overset{2}{1}, \overset{3}{0}, \overset{4}{0}, \overset{5}{1}, \cdots).$$

Constructing dyadic and 2-adic multipliers

Let us construct first a collection of multiplier functions which are the dyadic and 2-adic counterparts of the classical sense multiplier functions.

Definition

For the dyadic field $(\mathbb{I}, \overset{\circ}{+}, \circ)$ let us define the function with parameters $a \in \mathbb{I}_1, k \geq 1$ defined as

$$F_{a,k}(t) := w_k(B_a(t) \overset{\circ}{+} t) \quad (t \in \mathbb{I}). \tag{7}$$

Definition

For the 2-adic field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ let us define the function with parameters $a \in \mathbb{I}_1, k \geq 1$ defined as

$$F_{a,k}(t) := v_k(B_a(t) \overset{\bullet}{-} t) \quad (t \in \mathbb{I}). \tag{8}$$

$F_{a,k}$ $(a \in \mathbb{I}_1, k \geq 1)$ are multipliers

We will use the same notation for these groups, as it does not cause any confusion.

Theorem

The above defined functions $F_{a,k}$ $(a \in \mathbb{I}_1, k \geq 1)$ are multipliers of groups $(\mathbb{I}, \overset{\circ}{+})$ and $(\mathbb{I}, \overset{\bullet}{+})$ correspondingly.

Proof: Using the character property of v_k and w_k we find:

$$F_{a_1 \triangle a_2, k}(t) = w_k(B_{a_1 \triangle a_2}(t) \overset{\circ}{+} t) =$$
$$= w_k \left(B_{a_1}(B_{a_2}(t)) \overset{\circ}{+} B_{a_2}(t) \right) w_k \left(B_{a_2}(t) \overset{\circ}{+} t \right) =$$
$$= F_{a_1,k}(B_{a_2}(t)) F_{a_2,k}(t)$$

$$F_{a_1 \triangle a_2, k}(t) = w_k(B_{a_1 \triangle a_2}(t) \overset{\circ}{+} t) = w_k \left(B_{a_1}(B_{a_2}(t)) \overset{\circ}{+} B_{a_2}(t) \right) w_k$$
$$= F_{a_1,k}(B_{a_2}(t)) F_{a_2,k}(t).$$

Unitary representation of the dyadic and 2-adic Blaschke groups

Definition

The **representation of the Blaschke group** on $L^2(\mathbb{I})$ generated by this multiplier is given by

$$(U_{a,k}f)(x) := F_{a^{-1},k}(x) \cdot (f \circ B_{a^{-1}})(x) \quad (a \in \mathbb{I}_1).$$

Thus, in the dyadic case, $(U_{a,k}f)(x) = w_k(B_a(x) \overset{\circ}{+} x)f(B_a(x))$, while in the 2-adic case $(U_{a,k}f)(x) = v_k(B_a(x) \overset{\bullet}{-} x)f(B_{a^{-1}}(x))$.

Theorem

The function collection $(U_a)_{a\in\mathbb{I}_1}$ given as above is a unitary representation of the Blaschke group on $L^2(\mathbb{I})$.

Proof-Part 1

We will show that $a \to U_{a,k}$ defined by the above formula is a multiplier representation of $(\mathcal{B}, \circ)$. First, we have to check whether

$$U_{a_1,k} \circ U_{a_2,k} = U_{a_1 \triangle a_2,k} \quad (a_1, a_2 \in \mathbb{I}_1,\ k \in \mathbb{Z})$$

holds. The multiplier property $F_{a_1 \triangle a_2,k}(x) = F_{a_1,k}(B_{a_2}(x))F_{a_2,k}(x)$ and property $B_a \circ B_b = B_{a \triangleleft b}$ $(a, b \in \mathbb{I}_1)$ imply that

$$\begin{aligned} U_{a_1,k}(U_{a_2,k}f)(t) &= U_{a_1,k}\left(F_{a_2^{-1},k}(t) \cdot f\left(B_{a_2^{-1}}(t)\right)\right) = \\ &= F_{a_1^{-1},k}(t) \cdot F_{a_2^{-1},k}(B_{a_1^{-1}}(t)) \cdot f\left(B_{a_2^{-1}}(B_{a_1^{-1}}(t))\right) = \\ &= F_{a_2^{-1} \triangle a_1^{-1},k}(t) \cdot f(B_{a_2^{-1} \triangle a_1^{-1}}(t)) = \\ &= [F_{(a_1 \triangle a_2)^{-1},k} \cdot f \circ B_{(a_1 \triangle a_2)^{-1}}](t) = (U_{a_1 \triangle a_2,k} f) \end{aligned}$$

Proof-Part 2

The considered representation is unitary in the Hilbert space $H^2(\mathbb{I})$ with respect to the inner product $\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{I}} f(t)\overline{g}(t)dt \ \ (f, g \in H^2(\mathbb{I}))$. Indeed, as the Blaschke functions are measure-preserving transformations, we get

$$\langle U_{a,k}f, U_{a,k}g\rangle = \frac{1}{2\pi}\int_{\mathbb{I}} f\left(B_{a^{-1}}(t)\right)\overline{g}\left(B_{a^{-1}}(t)\right)dt = \\ = \langle f, g\rangle.$$

The voice transform

The **voice transform** of $f \in H$ generated by the representation $U_{\cdot,k}$ and by the parameter $\rho \in H$ is the (complex- valued) function on $\mathbb{I}_1$ defined by

$$(V_\rho f)(x) := \langle f, U_{x,k}\rho \rangle \quad (x \in \mathbb{I}_1, f, \rho \in H = L^2(\mathbb{I})).$$

Corollary

If $(U_x)_{x\in\mathbb{I}_1}$ is a unitary multiplier representation of $\mathbb{I}_1$ on $L^2_m(\mathbb{I})$ generated by $F_a \in L^\infty_m(\mathbb{I}) \cap C(\mathbb{I})$ $(a \in \mathbb{I}_1)$, then
$V_\rho \circ U_a = L_a \circ V_\rho$;
$V_\rho \circ L_a = L_a \circ V_\rho \circ M_a$;
$(V_\rho f)(x) = \overline{(V_\rho f)(x^{-1})}$ $(a, x \in \mathbb{I}, \rho \in H)$.

In the definition of the complex discrete Laguerre functions we find the trigonometric system. The functions corresponding to the trigonometric system $(e^{ikt}, k \in \mathbb{Z})$ $(t \in \mathbb{R})$ will be now the characters of the group $(\mathbb{I}, \overset{\circ}{+})$, namely the Walsh-Paley functions $(w_k, k \in \mathbb{N})$ presented in (**??**).
The dyadic discrete Laguerre functions associated to B_a with parameter $a \in \mathbb{I}_1$ by

$$L_k^{(a)}(x) := w_k(B_a(x)) \quad (k \in \mathbb{N}, x \in \mathbb{I}). \tag{9}$$

For $a \in \mathbb{I}_1$ consider the functions $r_n \circ B_a \ \ (x \in \mathbb{I}, \ n \in \mathbb{N})$.
(Here $\circ$ stands for function-composition.)
The dyadic discrete Laguerre system $(L_k^{(a)}, k \in \mathbb{N})$ is the product system generated by $(r_n \circ B_a, n \in \mathbb{N})$:

$$L_k^{(a)}(x) = \prod_{n=0}^{\infty} \left[r_n(B_a(x)) \right]^{k_n} .$$

As before, the functions corresponding to the orthonormed system $(e^{ikt}, k \in \mathbb{Z}, t \in \mathbb{R})$ will be the characters of the group $(\mathbb{I}, \dot{+})$, namely the functions $(v_k, k \in \mathbb{N})$ presented in (**??**). The *2-adic discrete Laguerre functions* associated to B_a are defined in the following way:

$$L_k^{(a)}(x) := v_k(B_a(x)) \qquad (k \in \mathbb{N}, x \in \mathbb{I}). \tag{10}$$

For $a \in \mathbb{I}_1$ and $n \in \mathbb{N}$ consider the functions $v_{2^n} \circ B_a$ on $\mathbb{I}$. The arithmetical discrete Laguerre system $(L_k^{(a)}, k \in \mathbb{N})$ is the product system generated by $(v_{2^n} \circ B_a, n \in \mathbb{N})$:

$$L_k^{(a)}(x) = \prod_{j=0}^{+\infty} [v_{2^j}(B_a(x))]^{k_j} \quad (x \in \mathbb{I}).$$

These form an orthogonal basis on $L^2(\mathbb{I})$ for all $a \in \mathbb{I}_1$.

Dyadic projection operator

Consider

$$V_{w_m} f(a^{-1}) = \langle f, U_{a^{-1},m} w_m \rangle.$$

Definition:

$$Pf(a,x) := \sum_{m=0}^{\infty} (V_{w_m} f)(a^{-1}) L_{a,m}(x) \quad (a \in \mathbb{I}_1,\ x \in \mathbb{I}),$$

where the infinite sequence is absolute convergent for $x \in \mathbb{I}$.

2-adic projection operator

Consider

$$V_{v_m} f(a^{-1}) = \langle f, U_{a^{-1},m} v_m \rangle.$$

Definition:

$$Pf(a, x) := \sum_{m=0}^{\infty} (V_{v_m} f)(a^{-1}) L_{a,m}(x) \quad (a \in \mathbb{I}_1,\ x \in \mathbb{I}),$$

where the infinite sequence is absolute convergent for $x \in \mathbb{I}$.

Conjecture:
For every $f \in H^2(\mathbb{S})$ and $a \in \mathbb{I}$, we have

$$\lim_{\|z\| \to 1} Pf(a, z) = f(x) \quad ?$$

Thank you for your attention!

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