Chebyshev Polynomials on Circular Arcs

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Chebyshev approximation problem on K ($K \subset \mathbb{C}$ compact):

$$\|\hat{\mathcal{P}}_N\|_K := \min\{\|\hat{P}_N\|_K : \hat{P}_N \in \hat{\mathbb{P}}_N\}$$

- $\hat{\mathcal{P}}_N(z)$ ... Chebyshev polynomial (of degree N) on K

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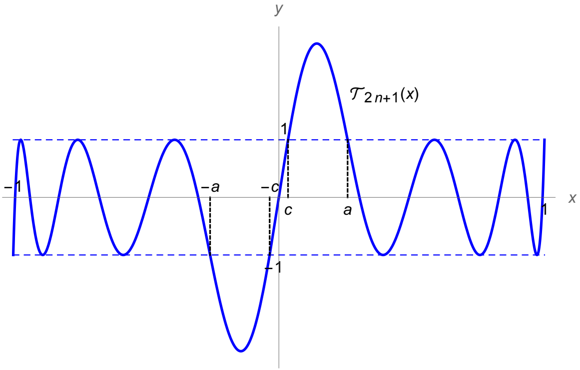
B. Eichinger (2017), based on results of Yuditskii:
Representation and asymptotics of $\hat{\mathcal{P}}_N(z)$ with complex Green function for $\overline{\mathbb{C}} \setminus A_\alpha$


y
x
$\mathcal{T}_{2n+1}(x)$
1
−1
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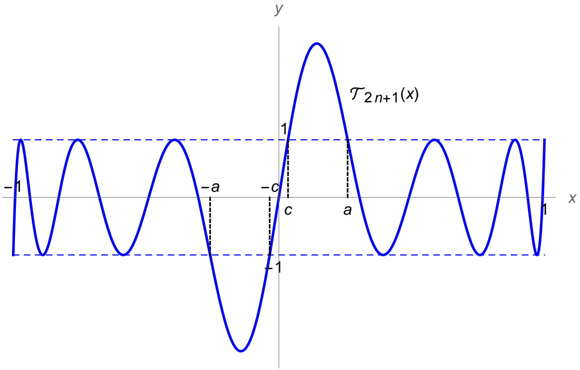
−c
c
a
−1
1



Abel-Pell equation:

$$\mathcal{T}_{2n+1}^2(x) + (1 - x^2)(x^2 - a^2)(x^2 - c^2)\,\mathcal{U}_{2n-2}^2(x) = 1$$

$$\hat{\mathcal{T}}_{2n+1}(x) := L\mathcal{T}_{2n+1}(x)$$

$$L = t_{2n+1}([-1, -a] \cup [a, 1])$$

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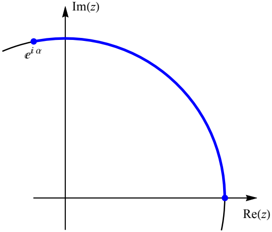

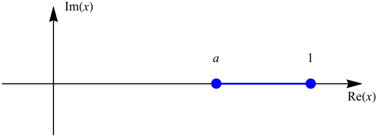

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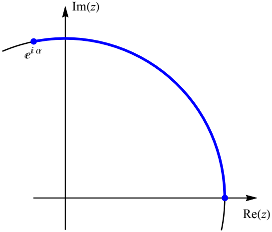

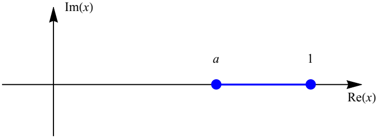

Detaille/Thiran 1991:

$$\mathrm{e}^{\mathrm{i}\varphi} \text{ e-point of } \hat{\mathcal{P}}_{2n}(z) \iff \cos\left(\tfrac{\varphi}{2}\right) \text{ e-point of } \mathcal{T}_{2n+1}(x)$$

$$\mathcal{T}_{2n+1}^2(x) + (1-x^2)(x^2-a^2)(x^2-c^2)\,\mathcal{U}_{2n-2}^2(x) = 1$$

Theorem (Sch 2019)

$$\hat{\mathcal{P}}_{2n}(z) = 2^{2n} L\, z^{n-1/2}\left(\mathcal{T}_{2n+1}(x) + \mathrm{i}\sqrt{1-x^2}\,(x^2-a^2)\,\mathcal{U}_{2n-2}(x)\right)$$

is the Chebyshev polynomial of degree $2n$ on A_α, where $x = \frac{1}{2}\left(\sqrt{z} + \frac{1}{\sqrt{z}}\right)$, with minimum norm

$$t_{2n}(A_\alpha) = 2^{2n} L$$

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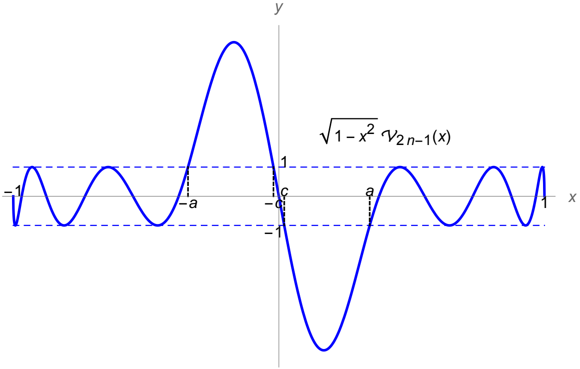
Proof with Kolmogorov criterion based on [Peherstorfer/Sch 2002].



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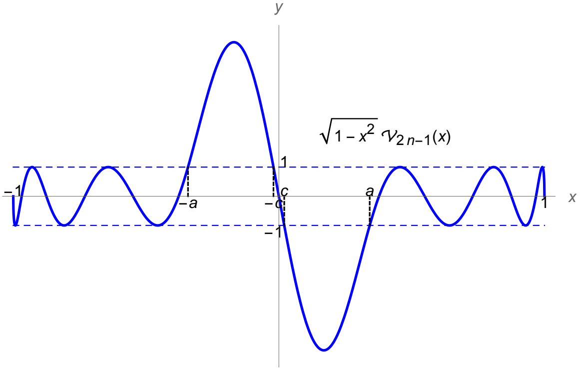

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$N = 2n$:

$$\hat{\mathcal{P}}_{2n}(z) = 2^{2n} L\, z^{n-1/2} \Big(\mathcal{T}_{2n+1}(x) + \mathrm{i}\sqrt{1-x^2}\,(x^2 - a^2)\,\mathcal{U}_{2n-2}(x) \Big)$$

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$\mathcal{T}_{2n+1}(x), \mathcal{U}_{2n-2}(x), \mathcal{V}_{2n-1}(x), \mathcal{W}_{2n-2}(x)$:
Representation with Jacobi's elliptic and theta functions

Parameter (Modulus) of the Jacobian elliptic and theta functions:

$$0 \leq k \leq 1$$

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Complete elliptic integral of the first kind:

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Jacobian theta functions (defined as Fourier series):

$$\mathrm{H}(u) \equiv \mathrm{H}(u,k),\ \ \mathrm{H}_1(u) \equiv \mathrm{H}_1(u,k),\ \ \Theta(u) \equiv \Theta(u,k),\ \ \Theta_1(u) \equiv \Theta_1(u,k)$$

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Jacobian elliptic functions:

$$\mathrm{sn}(u) := \frac{1}{\sqrt{k}} \cdot \frac{\mathrm{H}(u)}{\Theta(u)}\ \ \mathrm{cn}(u) := \frac{\sqrt{k'}}{\sqrt{k}} \cdot \frac{\mathrm{H}_1(u)}{\Theta(u)}\ \ \mathrm{dn}(u) := \sqrt{k'} \cdot \frac{\Theta_1(u)}{\Theta(u)}$$

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$$\mathrm{sn}(u) = \sin(u) \text{ (for } k \to 0) \qquad \mathrm{sn}(u) = \tanh(u) \text{ (for } k \to 1)$$

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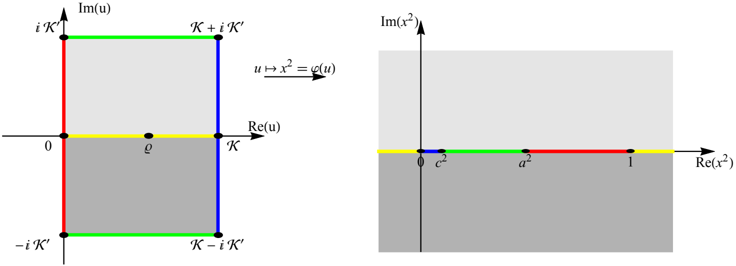

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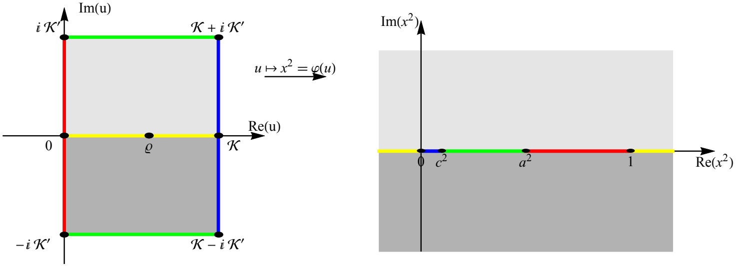

Compute k from $\mathrm{sn}(\varrho) = a$ where $\varrho := \frac{1}{N+1}\,\mathcal{K}$

$$x^2 = \varphi(u) := \frac{a^2\left(1 - \mathrm{sn}^2(u)\right)}{a^2 - \mathrm{sn}^2(u)}$$

$$\Omega(u) := \sqrt{\frac{\mathrm{H}(\varrho + u)}{\mathrm{H}(\varrho - u)}}$$

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Theorem (Sch 2019, Akhiezer 1928)

$$\mathcal{T}_{2n+1}(x) = \frac{1}{2}\left(\Omega(u)^{2n+1} + \Omega(u)^{-(2n+1)}\right) = \mathrm{Re}\left\{\Omega(u)^{2n+1}\right\}$$

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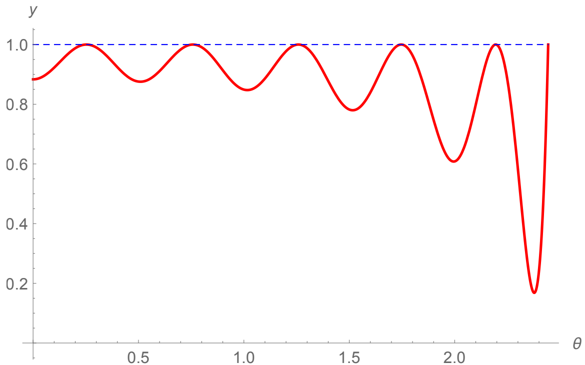
Plot of $y = |\mathcal{P}_{2n-1}(e^{i\theta})|^2$ for $2n - 1 = 11$,
$\alpha = 140^\circ \equiv \frac{7}{9}\pi = 2.4434\ldots$ and $0 \leq \theta \leq \alpha$



Results are valid also for two arcs (with the same length):

$$A_{\alpha,\beta} := \left\{ \mathrm{e}^{\mathrm{i}\theta} : \theta \in [-\alpha, -\beta] \cup [\beta, \alpha] \right\}$$

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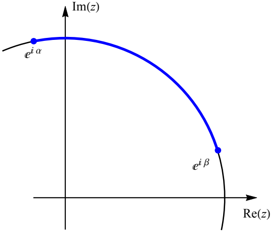

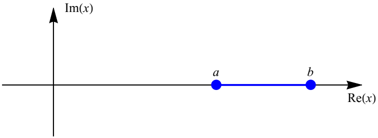

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Theorem (Sch 2019, Akhiezer 1956 without proof)

$$t_N(A_\alpha) \sim \left(\sqrt{1-a^2}\right)^{N+1} \cdot \sqrt{\frac{1+a}{1-a}} \qquad \textit{as } N \to \infty$$

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Theorem (Sch 2019, Akhiezer 1956 without proof)

$$t_N(A_\alpha) \sim \left(\sqrt{1-a^2}\right)^{N+1} \cdot \sqrt{\frac{1+a}{1-a}} \qquad \text{as } N \to \infty$$

Proof: Note that $N \to \infty \iff k \to 1$. Using results of [Carlson/Todd 1983], we proved that

$$t_N(A_\alpha) = \|\hat{\mathcal{P}}_N\|_{A_\alpha} = \left(\frac{\Theta(0)\,\Theta_1(0)}{\Theta(\varrho)\,\Theta_1(\varrho)}\right)^{N+1}$$

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$$\frac{\Theta(u)}{\Theta(0)} = \frac{\cosh(u)\cdot\exp\left(\mathcal{O}(k'^2\mathcal{K})\right)}{\exp\left(\frac{u^2}{2\mathcal{K}}\right)} \qquad \text{as } k \to 1$$

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See also Eichinger (2017).

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Conjecture:

$$\max\{1, 2\cos(\tfrac{\alpha}{2})\} \leq \frac{t_N(A_\alpha)}{\mathrm{cap}(A_\alpha)^N} \leq 1 + \cos(\tfrac{\alpha}{2})$$

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Thank you for your attention!