Sumsets and the Prékopa-Leindler inequality

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Joint work in (slow) progress
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Induced doubling

G commutative torsionfree group ($\mathbb{Z}^d$)
$V \subset G$ finite

Definition

$$\alpha''(V) = \inf_{A \subset G, A \supset V} \frac{|A+A|}{|A|}$$

the *induced doubling* of V

Examples: S_d the d-dimensional simplex, $\alpha''(S_d) = 1 + d/2$
$K_d = \{0,1\}^d$, discrete cube: $\sqrt{2}^d \leq \alpha''(K_d) \leq (3/2)^d$
(Green-Tao)

Variants

Two sets better than one set repeated:

Definition

The *induced doublings* of U are the quantities

$$\alpha(U) = \inf_{A \supset U, B \supset U} \frac{|A+B|}{\sqrt{|A||B|}},$$

$$\alpha'(U) = \inf_{A \supset U, B \supset U,\ |A|=|B|} \frac{|A+B|}{|A|},$$

$$\alpha''(U) = \inf_{A \supset U} \frac{|A+A|}{|A|},$$

the *unrestricted, isometric, isomeric induced doubling*.

Conjecture

$\alpha(U) = \alpha'(U) = \alpha''(U)$.

We know only $\alpha(U) \le \alpha'(U) \le \alpha''(V) \le \alpha(U)^2$.

Tripling

3 sets better than 2

Definition

The *triplings* of U are the quantities

$$\beta(U) = \inf_{A,B} \frac{|A+B+U|}{\sqrt{|A||B|}},$$

$$\beta'(U) = \inf_{A,B,\ |A|=|B|} \frac{|A+B+U|}{\sqrt{|A||B|}},$$

$$\beta''(U) = \inf_{A} \frac{|A+A+U|}{|A|},$$

the *unrestricted*, *isometric*, *isomeric tripling*.

Examples: $K_d = \{0,1\}^d$, discrete cube:
$\beta(K_d) = \beta'(K_d) = \beta''(K_d) = 2^d$.
S_d the d-dimensional simplex,
$\beta(S_d) = \beta'(S_d) = \beta''(S_d) = d + 1$

Conjecture

$\beta(U) = \beta'(U) = \beta''(U)$.

We know only $\beta(U) \leq \beta'(U) \leq \beta''(V) \leq \beta(U)^2$.

Tripling and induced doubling

Theorem

$\alpha(U) \leq \beta(U) \leq \alpha(U)^2$.

Problem

How tight are these inequalities?
We have $\beta(K_d) = 2^d$, $2^{d/2} \leq \alpha(K_d) \leq (3/2)^d$, so the second inequaliy is pretty tight, definitely no lower exponent than $\log 2/\log(3/2)$.
For a large cube $K = \{1, 2, \ldots, n\}^d$ we know $\beta(K) = 2^d$, $\alpha''(K) > 2^d - \varepsilon$, so if $\alpha = \alpha''$, then the first is tight.

Weights: sumset and max-convolution

Functions better than sets.
Consider functions $f : G \to \mathbb{R}_{\geq 0}$ with finite support.

Definition

The *max-convolution* of two functions is

$$f \mathbin{\overline{*}} g(x) = \max_t f(t)g(x-t).$$

Generalizes addition of sets: $\mathbf{1}_A \mathbin{\overline{*}} \mathbf{1}_B = \mathbf{1}_{A+B}$.
$\|\mathbf{1}_A\|_c^c = |A|$ for any $c > 0$.
Generalized cardinality: the *distribution function* of f is

$$F(t) = |\{x : f(x) > t\}|$$

If f, g have the same distribution, write $f \sim g$.
$\mathbf{1}_A \sim \mathbf{1}_B$ iff $|A| = |B|$.

Functional tripling

Definition

The *triplings* of a function f are

$$\gamma(f) = \inf_{g,h} \frac{\|f \mathbin{\overline{*}} g \mathbin{\overline{*}} h\|_1}{\|g\|_2 \|h\|_2}.$$

$$\gamma'(f) = \inf_{g\sim h} \frac{\|f \mathbin{\overline{*}} g \mathbin{\overline{*}} h\|_1}{\|g\|_2 \|h\|_2},$$

$$\gamma''(f) = \inf_{g} \frac{\|f \mathbin{\overline{*}} g \mathbin{\overline{*}} g\|_1}{\|g\|_2^2},$$

the *unrestricted*, *isometric*, *isomeric tripling*.

Conjecture

$$\gamma = \gamma' = \gamma''.$$

Tripling and functional tripling

Theorem

Let U be any finite set in a commutative group. We have

$$\beta(U) = \gamma(\mathbf{1}_U),\ \beta'(U) = \gamma'(\mathbf{1}_U),\ \beta'(U) = \gamma''(\mathbf{1}_U).$$

Multiplicativity of functional tripling

Let $G = G_1 \times G_2$, $f_i : G_i \to \mathbb{R}_{\geq 0}$, define $f : G \to \mathbb{R}_{\geq 0}$ by $f(x, y) = f_1(x)f_2(y)$.

Theorem

$\gamma(f) = \gamma(f_1)\gamma(f_2)$.

Corollary

If $V_i \subset G_i$, $V = V_1 \times V_2$, then $\beta(V) = \beta(V_1)\beta(V_2)$.

In particular, $\beta(K_d) = \beta(\{0, 1\})^d = 2^d$.

Tripling and functional tripling: proof

We prove $\beta(U) = \gamma(\mathbf{1}_U)$.
Recall

$$\beta(U) = \inf_{A,B} \frac{|A+B+U|}{\sqrt{|A||B|}} = \inf_{A,B} \frac{\|\mathbf{1}_U \mathbin{\overline{*}} \mathbf{1}_A \mathbin{\overline{*}} \mathbf{1}_B\|_1}{\|\mathbf{1}_A\|_2 \|\mathbf{1}_B\|_2}$$

$$\gamma(\mathbf{1}_U) = \inf_{g,h} \frac{\|\mathbf{1}_U \mathbin{\overline{*}} g \mathbin{\overline{*}} h\|_1}{\|g\|_2 \|h\|_2}.$$

Clearly $\beta(U) \geq \gamma(\mathbf{1}_U)$.
Need to prove $\beta(U) \leq \gamma(\mathbf{1}_U)$, that is,

$$\|\mathbf{1}_U \mathbin{\overline{*}} g \mathbin{\overline{*}} h\|_1 \geq \beta(U) \|g\|_2 \|h\|_2 .$$

Proof of $\|\mathbf{1}_U \overline{*} g \overline{*} h\|_1 \geq \beta(U) \|g\|_2 \|h\|_2$

Assume $\max g = \max h = 1$. Put $f = \mathbf{1}_U \overline{*} g \overline{*} h$. Let

$$\mathcal{F}(t) = \{x : f(x) \geq t\}, \quad t \in [0,1]; \; F(t) = |\mathcal{F}(t)|,$$

similarly $\mathcal{G}(t), G(t), \mathcal{H}(t), H(t)$ from g, h. For $t = xy$

$$\mathcal{F}(t) \supset \mathcal{G}(x) + \mathcal{H}(y) + U$$

$$F(t) \geq \beta(U) \max_{xy=t} \sqrt{G(x)H(y)}.$$

$$\|f\|_1 = \int_0^1 F(t)\, dt \geq$$

$$\geq \beta(U) \int_0^1 \max_{xy=t} \sqrt{G(x)H(y)}\, dt = ?$$

Lemma

$$\int_0^1 \max_{xy=t} \sqrt{G(x)H(y)}\, dt \geq$$

$$\geq \sqrt{\int_0^1 G(\sqrt{t})\, dt \int_0^1 H(\sqrt{t})\, dt} = \|g\|_2 \, \|h\|_2 \, .$$

Lemma (Prékopa-Leindler inequality (special case))

If always

$$\varphi\left(\frac{x+y}{2}\right) \geq \sqrt{\mu(x)\nu(x)}$$

then $\|\varphi\|_1 \geq \sqrt{\|\mu\|_1 \|\nu\|_1}$.

To get the needed lemma, apply Prékopa-Leindler with

$$\mu(x) = G(e^x)e^{2x}, \quad \nu(x) = H(e^x)e^{2x}$$

$$\varphi(t) = e^{2t} \max_{x+y=t} \sqrt{G(e^{2x})H(e^{2y})}.$$

Generalized direct product

Let $G = G_1 \times G_2$, $V_1 \subset G_1$, for each $v \in V_1$ given a $W_v \subset G_2$ and

$$V = \bigcup_{v \in V_1} v \times W_v.$$

Theorem

$$\beta(V) \geq \beta(V_1) \min b(W_v).$$

Theorem (functional version)

Let f be a function on G. For $x \in G_1$ put $f_x(y) = f(x, y)$, a function on G_2, and $\varphi(x) = \gamma(f_x)$.

$$\gamma(f) \geq \gamma(\varphi).$$

Starting point

Theorem

Let f be a function supported on 2 points. Then $\gamma(f) = \|f\|_1$.

Generalized cube

A *generalized cube* is obtained from 2-element sets by generalized direct product.
1-dimensional generalized cube: any 2-element set.
$d+1$ -dimensional g.c.: a set homothetic to
$(0 \times V) \cup (1 \times W)$, where V, W are d -dimensional g. cubes.
(Fairly general 2^d -element d-dimensional set.)
E.g. 2-dimensional g.c.: trapezoid.

Theorem

Let U be a d-dimensional quasicube. For every $V \subset U$ we have

$$\beta(V) = |V|, \quad \alpha(V) \geq |V|^{1/2}.$$

In particular

$$\beta(U) = 2^d, \quad \alpha(U) \geq 2^{d/2}.$$

A Brunn-Minkowski variation

Theorem

Let U be a d-dimensional quasicube. For any finte sets A, B we have

$$|A + B + U|^{1/d} \geq |A|^{1/d} + |B|^{1/d}.$$

Dependence matroid

Conjecture

Let V be finite set with the property that for any $k \leq \dim V$ and k-dimesional subset of V has at most 2^d elements. We have

$$\beta(V) = |V|, \quad \alpha(V) \geq |V|^{1/2}.$$

Conjecture

Let U, V be finite sets of equal cardinality, $\varphi : U \to V$ a bijection. If for every $U' \subset U$ we have $\dim \varphi(U') \leq \dim U'$, then $\beta(V) \leq \beta(U)$. In particular, if always $\dim \varphi(U') = \dim U'$, then $\beta(V) = \beta(U)$.

Continuity of α, β, γ

Problem

Are α, β, γ upper semicontinuous?
If $A = \{a_1, \ldots, a_n\} \subset \mathbb{Q}^d$, is there an $\varepsilon > 0$ such that whenever $A' = \{a_1', \ldots, a_n'\} \subset \mathbb{Q}^d$, $|a_i - a_i'| < \varepsilon$, then $\alpha(A') \geq \alpha(A)$ etc.?

The End