AN APPROXIMATE FORMULA FOR GOLDBACH'S PROBLEM WITH APPLICATIONS

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Notation: $p, p_i, p' \in \mathcal{P}$ primes, $m, n \in \mathbb{Z}^+$ $L(s, \chi)$ Dirichlet's L-function

$\mathcal{E}(X) = \{n \leq X;\ 2 \mid n,\ n \neq p + p'\}$, $|\mathcal{E}(X)| = E(X)$, $L = \log X$

(Binary) Goldbach Conjecture (BGC) (1742): $E(X) = 1$ if $X > 2$

(Ternary) Goldbach Conjecture (TGC): if $n > 5$, $2 \nmid n \Rightarrow n = p_1 + p_2 + p_3$

Landau (1912): "unattackable" at the present state of science

Hardy–Littlewood (1923–24): If $\mathcal{L}(s, \chi) \neq 0$ for $\mathrm{Re}\ s > \frac{3}{4}$, then TGC is true for $n > n_0$, $2 \nmid n$.

HL (1924) GRH $\Rightarrow E(X) = O(X^{1/2+\varepsilon})$ for $\forall \varepsilon > 0$.

Circle method: Let $m \in [X/2, X]$, $P < \sqrt{X}$, $Q = X/P$

Major arcs $\mathfrak{M} = \bigcup_{q \le P} \bigcup_{\substack{a \\ (a,q)=1}} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right]$

$\mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] - \mathfrak{M}$, $X_1 = X^{1-\varepsilon_0}$,

$S(\alpha) = \sum_{X_1 < p \le X} \log p e(p\alpha)$

$R(m) = \sum_{\substack{p+p'=m \\ p,p' \ge X_1}} \log p \cdot \log p' = R_1(m) + R_2(m)$

$R_1(m) = \int_{\mathfrak{M}} S^2(\alpha) e(-m\alpha) d\alpha$, $R_2(m) = \int_{\mathfrak{m}} S^2(\alpha) e(-m\alpha) d\alpha$

Traditional attack (HL, Vinogradov): If P is "sufficiently small" compared to X, then

$$R_1(m) \sim \mathfrak{S}(m)m \quad \text{where}$$

$$\mathfrak{S}(m) = \prod_{p|m}\left(1 + \frac{1}{p-1}\right)\prod_{p\nmid m}\left(1 - \frac{1}{(p-1)^2}\right).$$

Try to estimate $R_2(m)$ as $|R_2(m)| < R_1(m) \Rightarrow R(m) > 0$, at least on average (for most values of $m \leq N$, or $m = N - p_i$, $p_i \in \mathcal{P}$).

HL (1923) could asymptotically evaluate $R_1(m)$ and estimate well $R_2(m)$ on average by using the still today unproved hypothesis $\mathcal{L}(s,\chi) \neq 0$ for $\mathrm{Re} s > \frac{3}{4}$. This led to a conditional solution of TGC for $n > n_0$.

Vinogradov (1937) – later Vaughan – proved

Theorem A:

$$S(\alpha) \ll \left(\frac{X}{\sqrt{P}} + X^{4/5} \right) L^c \quad \textit{if} \quad \alpha \in \mathfrak{m}.$$

Theorem B: $R_1(m) \sim \mathfrak{S}(m)m$ *holds for* $P = L^A$ for $\forall$ fixed A.

This was a Corollary of Siegel's theorem (1936).

Theorem A implies that $|R_2(m)| \ll \frac{X}{\sqrt{P}} L^c$ on average, namely

$$\sum_{m=X/2}^{X} R_2^2(m) = \int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \le \Big(\max_{\alpha\in\mathfrak{m}} |S(\alpha)|\Big)^2 \int_0^1 |S(\alpha)|^2 d\alpha$$

$$\ll \max\left(\frac{X^2}{P}, X^{8/5}\right) X L^c$$

Theorem C (Vinogradov): *TGC is true for* $n > n_0$.

Estermann, Van der Corput, Cudakov (1937–38):

$$E(X) \ll_A \frac{X}{(\log X)^A} \quad \text{for} \quad \forall A > 0 \quad (\Rightarrow \text{ TGC}).$$

This implies the existence of inf. many 3-term AP's in $\mathcal{P}$ (Van der Corput) since $E(X) = o(\pi(X))$.

Theorem D (Helfgott, 2013): *TGC is true for* $n > 5$, $2 \nmid n$.

Goal (in view of Theorem A): to increase P in such a way that $|R_2(m)| < R_1(m)$ should hold for $m \in [X/2, X]$ apart from a "small" exceptional set $\mathcal{E}(X)$.

Theorem E (Montgomery–Vaughan, 1975): $E(X) < X^{1-c_0}$ *with a not calculated but effective small* $c_0 > 0$*.*

Main idea: To evaluate $R_1(m)$ taking into account the effect of the possibly existing single Siegel zero + use of Gallagher's theorem for primes in AP.

Choice of P: $P = X^{c_1}$ ($c_1 > 0,\ \ $ small)

Chen (1989) $c_0 = 0.05$ (proof is incorrect).

Hongze Li (2000) $c_0 = 0.086 \iff E(X) \ll X^{0.914}$.

Theorem F (Wen Chao Lu, 2010):
$c_0 = 0.121 \iff E(X) \ll X^{0.879}$.

Main advantages:

(i) P can be chosen quite large, $P = X^{4/9-\varepsilon} \Rightarrow$

(ii) estimates on the minor arc will be much better

(iii) it gives a characterisation of the possible exceptional m values (in terms of some "bad primitive characters" and their conductors) $\Longleftrightarrow$

(iv) for a fixed m it determines the conductors of possible "bad primitive characters" which might cause $m \in \mathcal{E}(X)$

Which ones are the "bad primitive characters"?

Those which have low zeros near to $\mathrm{Re}\ s = 1$.

Definition: $\mathcal{E} = \mathcal{E}(H, T, P, X)$ the set of generalized exceptional singularities of all primitive $\dfrac{\mathcal{L}'}{\mathcal{L}}$ functions $\bmod r$, $r \le P$ $(\chi_0 = \chi_0 (\mathrm{mod}\ 1))$

$(\varrho_0, \chi_0) \in \mathcal{E}$ if $\varrho_0 = 1$

$(*)$

$$(\varrho_i, \chi_i) \in \mathcal{E} \text{ if } \exists \chi_i \text{ primitive, } \mathrm{cond}\ \chi_i = r_i \le P, \quad L(\varrho_i, \chi_i) = 0,$$

$$\beta_i \ge 1 - H/\log X, \quad |\gamma_i| \le T \quad (\varrho = \beta + i\gamma)$$

$$A(\varrho) = 1 \text{ if } \varrho = 1, \quad A(\varrho) = -1 \text{ if } \varrho \ne 1$$

Definition: $\mathfrak{S}(\chi_i, \chi_j, m)$ generalized exceptional singular series

$$\mathfrak{S}(\chi_0, \chi_0, m) := \mathfrak{S}(m) = \prod_{p|m}\left(1 + \frac{1}{p-1}\right)\prod_{p\nmid m}\left(1 - \frac{1}{(p-1)^2}\right).$$

Theorem 1. Let $\varepsilon < \varepsilon_0$ and $\varepsilon < \vartheta < 4/9 - \varepsilon$ be fixed, $m \in [X/2, X]$ $\exists P \in (X^{\vartheta-\varepsilon}, X^{\vartheta})$ such that for $X > X_0(\varepsilon)$
(1)

$$R_1(m) = \sum_{\varrho_i \in \mathcal{E}} \sum_{\varrho_j \in \mathcal{E}} A(\varrho_i)A(\varrho_j)\mathfrak{S}(\chi_1, \chi_2, m)\frac{\Gamma(\varrho_i)\Gamma(\varrho_j)}{\Gamma(\varrho_i + \varrho_j)} m^{\varrho_i + \varrho_j - 1} + O_\varepsilon\big(\mathfrak{S}(m)Xe^{-c_0 H}\big) + O_\varepsilon(X^{1-\varepsilon_0})$$

Main lemma. $|\mathfrak{S}(\chi_1, \chi_2, m)| \leq \mathfrak{S}(m)$ always,

further $|\mathfrak{S}(\chi_1, \chi_2, m)| \leq \dfrac{\mathfrak{S}(m)}{\sqrt{U}} \log_2^2 U$, where

$$U = U(\chi_1, \chi_2, m) = \max\left(\frac{r_1}{(m, r_1)}, \frac{r_2}{(m, r_2)}, \ \text{cond } \chi_1\chi_2\right)$$

Remark 1 (follows from a Theorem of Jutila). The total number of characters in $(*)$ is

$$K \leq C_1 e^{2H} \tag{2}$$

Further we have $r_i \gg 4^2$.

Summary: Choosing H and T large constants, the total number of zeros in (1) will be bounded and their contribution will be negligible ($O(\varepsilon)$) *unless*

$$(3) \quad |\gamma_i| \le T, \text{ cond } \chi_1\chi_2 \le C(\varepsilon), \quad r_i \mid C(\varepsilon)m \quad (i = 1, 2)$$

Remark 2. Siegel zeros cause a lot of trouble but the case of their existence can be handled by an improved form of the Deuring–Heilbronn phenomenon (J. P. 2019).

So, suppose in the following that they do not exist.

Let m be given, r_i $(0 \leq i \leq K)$ cond. of gen. exc. char.

Let $\mathcal{K}_m = \{0 \leq i \leq K;\ r_i \mid C(\varepsilon)m\}$
$K(m) = \mathrm{l.c.m.}\ [r_i, i \in \mathcal{K}_m]$.

Case 1. If $K(m) > P$, then the number of such $m \in [X/2, X]$ is $\ll_\varepsilon \dfrac{X}{P}$.

Case 2. If $K(m) \leq P$, then l.c.m. $[r_i, r_j \in \mathcal{K}_m] \leq K(m) \leq P$ so there exists a q (depending on $\mathcal{K}_m$) such that all χ_i are (may be not primitive) characters $\bmod q$.

Corollary.

Remark 3: If $A = \log X / \log P$ then $X^{-(\delta_1+\delta_2)} \leq e^{-A(\delta_1+\delta_2)}$.

Theorem 2. (3) *is true if* $q \leq P = X^{7/25} = X^{0.28}$.

Remark 4. The proof of Theorem 2 uses a series of new density theorems (extremely) near to $\mathrm{Re}\ s = 1$.

This implies (using Theorem 1 and Theorem A)

Theorem 3. $E(X) \ll X^{0.72}$.

1.) The Linnik–Goldbach problem

Theorem (Linnik 1951, 1953). *Every sufficiently large even number can be written as the sum of two primes and K powers of two.*

$K = 54\,000$, GRH $\Rightarrow K = 770$ [Liu–Liu–Wang, 1998]

$K = 25\,000$ (Hongze Li, 2000), GRH $\Rightarrow K = 200$ [LLW, 1999]

$K = 2250$, GRH $\Rightarrow K = 160$ (Wang, 1999)

$K = 1906$ (Hongze Li, 2001)

Announcement (J. P., Debrecen, 2000) $K = 12$, GRH
$\Rightarrow K = 10$

$K = 13$, GRH $\Rightarrow K = 7$ (Heath-Brown and Puchta, 2002)

(J. P. – Ruzsa, 2003) GRH $\Rightarrow K = 7$

Elsholtz $K = 12$

Theorem 4 (J. P. – Ruzsa). $K = 8$ *unconditionally.*

Main idea (beyond the work showing GRH $\Rightarrow K = 7$).

If the GRH is true, we can basically take $P = \sqrt{X}$.

How can we get so close to the conditional result $K = 7$?

Answer: The explicit formula allows us to take $P = X^{4/9-\varepsilon}$ which implies e.g. $S(\alpha) \ll X^{4/5} L^c$ for $\alpha \in \mathfrak{m}$.

Question: What happens on the major arcs?

Remark 5: We can not guarantee $R_1(m) > 0$ for all m.

Crucial point. Since – as mentioned in the summary to the explicit formula – we know the "*structure of the possible exceptional set concerning the major arcs*", namely, *a bounded number of bad moduli and their multiples* we have

Theorem 5. *Under the above conditions*

$$\sum_{\nu \le L} R_1(m - 2^\nu) = (1 + O(\varepsilon)) \sum_{\nu \le L} \mathfrak{S}(m - 2^\nu)(m - 2^\nu). \tag{4}$$

2.) Goldbach numbers in thin sequences

$$\mathcal{E}_k(N) = \left\{ n \le N;\ 2n^k \ne p + p' \right\} \quad |\mathcal{E}_k(N)| = E_k(N)$$

Are almost all numbers of the form $2n^k$ Goldbach?

Perelli 1996: Yes, $E_k(N) \ll_A N(\log N)^{-A}$ for $\forall A$ fixed.

Brüdern, Kawada, Wooley 2000: $E_k(N) \ll N^{1-c/k}$.

Here c is a very small absolute constant (depending on a crucial constant in Gallagher's theorem).

Remark 6: The strongest known estimate $E(X) \ll X^{1/2+\varepsilon}$ under GRH (HL 1924) gives $E_2(N) \ll N^{1+\varepsilon}$ which is worse than the trivial estimate.

Let us consider the special case $k = 2$.

BKW: $E_2(N) \ll N^{1-c_1}$ $c_1 > 0$ small, not calculated.

Theorem 6 (A. Perelli and J. P.): $E_2(N) \ll N^{4/5+\varepsilon}$.

Remark 7: We can show similar explicit results for small k ($k = 3, 4, 5, \dots$), further for $k \to \infty$ we can show

$$E_k(N) \ll N^{1-1/(5+\varepsilon)k} \quad \text{for} \quad k > k_0(\varepsilon).$$

Key ideas: (i) one can choose $P \in [X^{0.4}, X^{0.41}] \Rightarrow$ approximate formula works (Theorem 1).

(ii) in contrast to the Linnik–Goldbach problem we need also Theorem 2 (actually a weaker form of it would be enough, namely with $X = N^2$

$$\sum_{\varrho_1(\chi_1,q)\in\mathcal{E}} \sum_{\substack{\varrho_2(\chi_2,q)\in\mathcal{E} \\ \mathrm{cond}(\chi_i\chi_j)<C(\varepsilon)}} X^{-\delta_1-\delta_2} < 1 - c_2(\varepsilon) \ \text{ if } \ q \le X^{1/10} = N^{1/5}$$

using the fact that $\dfrac{r_i}{(r_i,4)}$ are squarefree so “essentially” $r_i \mid n \Longleftrightarrow r_i \mid n^2$

(iii) in the minor arcs we use the nice method of BKW using the Vinogradov–Vaughan's estimate for $S(\alpha)$ and Weyl's inequality for estimating exponential sums over k-th powers

(iv) for $k > 2$ we use the deep estimates of Wooley and Ford–Wooley.