On the Borel summability of WKB solutions near a simple pole

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Schrödinger-type equation with a simple pole

Consider the second-order linear differential equation

$$\frac{\mathrm{d}^2 w(z,u)}{\mathrm{d}z^2} = (u^2 f(z) + g(z))w(z,u),$$

where u is a large positive parameter and $f(z)$ and $g(z)$ are analytic in a domain $\mathcal{G}$ except at a point $z_* \in \mathcal{G}$. In the neighbourhood of this point, assume that

$$f(z) = \frac{f_{-1}}{z - z_*} + f_0 + f_1(z - z_*) + \cdots,$$

$$g(z) = \frac{g_{-2}}{(z - z_*)^2} + \frac{g_{-1}}{z - z_*} + g_0 + g_1(z - z_*) + \cdots,$$

with $f_{-1} \neq 0$ and $g_{-2} = \frac{1}{4}(\mu^2 - 1)$, $\mu \in \mathbb{C} \setminus \mathbb{Z} + \frac{1}{2}$. We also assume that there are no zeros of $f(z)$ in $\mathcal{G}$.

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Transformation into normal form

With the Liouville transformation

$$\xi \stackrel{\text{def}}{=} \xi(z) \stackrel{\text{def}}{=} \int_{z_*}^{z} \sqrt{f(t)}\,\mathrm{d}t, \quad W(\xi, u) \stackrel{\text{def}}{=} f^{1/4}(z) w(z, u),$$

the equation becomes

$$\frac{\mathrm{d}^2 W(\xi, u)}{\mathrm{d}\xi^2} = \left(u^2 + \psi(\xi)\right) W(\xi, u),$$

where

$$\psi(\xi) \stackrel{\text{def}}{=} \frac{g(z)}{f(z)} + \frac{4f(z)f''(z) - 5f'^2(z)}{16 f^3(z)}.$$

Note that the function $\psi(\xi)$ is even and has a double pole at $\xi = 0$.

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WKB solutions

It is known that if $\psi(\xi)$ is analytic and bounded in a domain $\mathcal{D}$, then the equation has solutions $W_{1,2}(\xi, u)$ such that

$$W_1(\xi, u) = \mathrm{e}^{u\xi}(1 + \eta_1(\xi, u)), \quad W_2(\xi, u) = \mathrm{e}^{-u\xi}(1 + \eta_2(\xi, u)),$$

where $\eta_1(\xi, u)$, $\eta_2(\xi, u) = \mathcal{O}(u^{-1})$ as $u \to +\infty$ and ξ lies in certain subdomains of $\mathcal{D}$.

With some further conditions on $\psi(\xi)$ and $\mathcal{D}$, we can identify the solutions uniquely by requiring

$$\lim_{\mathfrak{Re}\,\xi \to -\infty} \mathrm{e}^{-u\xi} W_1(\xi, u) = 1$$

and

$$\lim_{\mathfrak{Re}\,\xi \to +\infty} \mathrm{e}^{u\xi} W_2(\xi, u) = 1.$$

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The asymptotics of $W_{1,2}(\xi,u)$ can be extended to asymptotic expansions:

$$W_1(\xi,u) \sim \mathrm{e}^{u\xi} \sum_{n=0}^{\infty} \frac{\mathsf{A}_n(-\xi)}{u^n},$$
$$W_2(\xi,u) \sim \mathrm{e}^{-u\xi} \sum_{n=0}^{\infty} \frac{\mathsf{A}_n(\xi)}{u^n},$$

as $u \to +\infty$ and ξ lies in appropriate domains. These are called the WKB solutions, named after the physicists Wentzel, Kramers and Brillouin. The coefficients satisfy the recurrence relation

$$\mathsf{A}_{n+1}(\xi) = \tfrac{1}{2}\mathsf{A}_n'(\xi) + \tfrac{1}{2}\int_{\xi}^{\infty} \psi(t)\mathsf{A}_n(t)\,\mathrm{d}t,$$

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Borel summation and connection formulae

We define

$$\mathcal{D}_{\theta_1,\theta_2}(\gamma) \stackrel{\text{def}}{=} \{\xi : |\mathfrak{Im}\xi| < \gamma,\, \theta_1 < \arg\xi < \theta_2,\, \theta_2 - \theta_1 \geq \pi\}\,.$$

Our first aim is to show that under certain conditions

$$W_1(\xi,u) = \mathrm{e}^{u\xi}\left(1 + \int_0^{+\infty} \mathrm{e}^{-ut} F_1(\xi,t)\,\mathrm{d}t\right),$$

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where $F_1(\xi,t)$ and $F_2(\xi,t)$ are analytic functions in domains of the form $\mathcal{D}_{\theta_1,\theta_2}(\gamma) \times \Sigma \subset \widehat{\mathbb{C}} \times \mathbb{C}$ apart from some simple singularities. Here $\widehat{\mathbb{C}}$ denotes the Riemann surface of the logarithm.

Our second aim is to determine the connection between these solutions as ξ crosses the Stokes rays. On a Stokes ray, it holds that $\mathfrak{Im}\xi = \mathfrak{Im}\int_{z_*}^{z}\sqrt{f(t)}\,\mathrm{d}t = 0$.

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Motivation: formal Borel summation

The idea behind our first aim is the following series of formal manipulations:

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W_2(\xi,u) = \mathrm{e}^{-u\xi}\sum_{n=0}^{\infty}\frac{\mathsf{A}_n(\xi)}{u^n} &= \mathrm{e}^{-u\xi}\left(1+\sum_{n=0}^{\infty}\frac{\mathsf{A}_{n+1}(\xi)}{u^{n+1}}\right)\\
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Remarks

Kamimoto and Koike studied the case when $f(z)$ is a meromorphic function with a simple pole at the origin and $g(z) \equiv 0$. From their analysis, the connection formula follows in a sufficiently small neighbourhood of the Stokes curve that emanates from the origin. The paper has not been published in a peer-reviewed journal yet but is available online. It relies on the works of Aoki, Kawai and Takei.

Koike and Schäfke were working on the Borel summability of WKB solutions on Stokes regions for Schrödinger-type equations with polynomial or rational potentials. Koike passed away last year and the paper has not been completed so far. An alternative proof for polynomial potentials was provided by Takei.

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Assume that $\psi(\xi)$ is analytic in $\{\xi : |\mathfrak{Im}\xi| < \gamma\} \setminus \{0\}$ with some $\gamma > 0$, and that there exist positive constants ρ and c such that

$$\left|\psi(\xi) - \frac{\mu^2 - \frac{1}{4}}{\xi^2}\right| \leq \frac{c}{1 + |\xi|^{1+\rho}},$$

when $\xi \in \{\xi : |\mathfrak{Im}\xi| < \gamma\} \setminus \{0\}$.

For any $0 < \varepsilon < \frac{\gamma}{2}$, define the domain Σ_ε of the complex plane via

$$\Sigma_\varepsilon \overset{\text{def}}{=} \left\{t : \exp\left(-\frac{\pi}{4\varepsilon}|\mathfrak{Re}t|\right) < 2\cos\left(\frac{\pi}{4\varepsilon}\mathfrak{Im}t\right), |\mathfrak{Im}t| < 2\varepsilon\right\} \cup \{t : |t| < 2\varepsilon\}.$$

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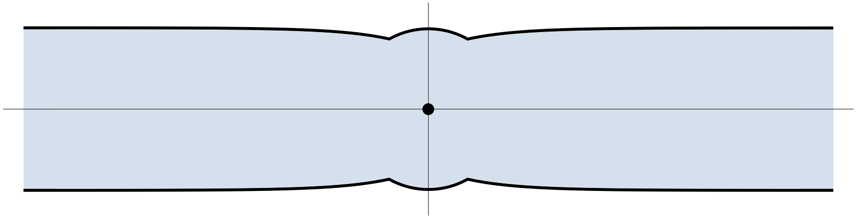
The domain Σ_ε



The domain Σ_ε in the t-plane.

Borel summability

Theorem (G. N., 2019)

Let $0 < \varepsilon < \frac{\gamma}{2}$ be arbitrary. Under our assumptions, we have, for any $u > 0$,

$$W_1(\xi, u) = \mathrm{e}^{u\xi} \left(1 + \int_0^{+\infty} \mathrm{e}^{-ut} F_1(\xi, t) \, \mathrm{d}t \right),$$

$$W_2(\xi, u) = \mathrm{e}^{-u\xi} \left(1 + \int_0^{+\infty} \mathrm{e}^{-ut} F_2(\xi, t) \, \mathrm{d}t \right).$$

If $2\xi \notin \Sigma_\varepsilon$ then $F_1(\xi, t)$ is analytic in $\mathcal{D}_{0,2\pi}(\gamma - \varepsilon) \times \Sigma_\varepsilon$ and if $2\xi \in \Sigma_\varepsilon$ then $F_1(\xi, t)$ has a simple singularity at $t = 2\xi$. Similarly, if $-2\xi \notin \Sigma_\varepsilon$ then $F_2(\xi, t)$ is analytic in $\mathcal{D}_{-\pi,\pi}(\gamma - \varepsilon) \times \Sigma_\varepsilon$ and if $-2\xi \in \Sigma_\varepsilon$ then $F_2(\xi, t)$ has a simple singularity at $t = -2\xi$. Finally, $F_2(\xi, t) = F_1(\xi \mathrm{e}^{\pi \mathrm{i}}, t)$.

We conjecture that Σ_ε can be replaced by the larger set $\{t : |\Im m\, t| < 2\varepsilon\}$.

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If $2\xi \notin \Sigma_\varepsilon$ then $F_1(\xi, t)$ is analytic in $\mathcal{D}_{0,2\pi}(\gamma - \varepsilon) \times \Sigma_\varepsilon$ and if $2\xi \in \Sigma_\varepsilon$ then $F_1(\xi, t)$ has a simple singularity at $t = 2\xi$. Similarly, if $-2\xi \notin \Sigma_\varepsilon$ then $F_2(\xi, t)$ is analytic in $\mathcal{D}_{-\pi,\pi}(\gamma - \varepsilon) \times \Sigma_\varepsilon$ and if $-2\xi \in \Sigma_\varepsilon$ then $F_2(\xi, t)$ has a simple singularity at $t = -2\xi$. Finally, $F_2(\xi, t) = F_1(\xi \mathrm{e}^{\pi \mathrm{i}}, t)$.

We conjecture that Σ_ε can be replaced by the larger set $\{t : |\mathfrak{Im}\, t| < 2\varepsilon\}$.

Connection formulae

Proposition (G. N., 2019)

Let ε be an arbitrarily small positive number. Then, under our assumptions, we have, for any $u > 0$,

$$W_1(\xi, u) = W_1(\xi e^{2\pi i}, u) + \frac{\cos(\pi\mu)}{2i} W_2(\xi, u)$$

provided $\xi \in \mathcal{D}_{-\pi,0}(\gamma - \varepsilon)$, and

$$W_2(\xi, u) = W_2(\xi e^{-2\pi i}, u) - \frac{\cos(\pi\mu)}{2i} W_1(\xi, u)$$

provided $\xi \in \mathcal{D}_{\pi,2\pi}(\gamma - \varepsilon)$.

Stokes phenomena

Corollary (G. N., 2019)

Let ε be an arbitrarily small positive number. Then, under our assumptions, the following asymptotic expansions hold as $u \to +\infty$:

$$W_1(\xi,u) \sim \mathrm{e}^{u\xi} \sum_{n=0}^{\infty} \frac{\mathsf{A}_n(-\xi)}{u^n} + \frac{\cos(\pi\mu)}{2\mathrm{i}} \mathrm{e}^{-u\xi} \sum_{n=0}^{\infty} \frac{\mathsf{A}_n(\xi)}{u^n}$$

provided $\xi \in \mathcal{D}_{-\pi,0}(\gamma - \varepsilon)$, and

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Example: Associated Legendre equation

For complex ν and μ, the associated Legendre equation takes the form

$$(z^2-1)\frac{\mathrm{d}^2L(z)}{\mathrm{d}z^2}+2z\frac{\mathrm{d}L(z)}{\mathrm{d}z}-\left(\nu(\nu+1)+\frac{\mu^2}{z^2-1}\right)L(z)=0.$$

Elimination of the first derivative yields

$$\frac{\mathrm{d}^2w(z,u)}{\mathrm{d}z^2}=(u^2f(z)+g(z))w(z,u),$$

with $u=\nu+\frac{1}{2}$ and

$$f(z)=\frac{1}{z^2-1},\quad g(z)=\frac{\mu^2-1}{4(z^2-1)^2}-\frac{1}{4(z^2-1)},$$

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The corresponding Liouville transformation is

$$\xi = \xi(z) = \int_1^z \frac{\mathrm{d}t}{\sqrt{t^2 - 1}} = \cosh^{-1} z.$$

This is a biholomorphic bijection between $\mathbb{C} \setminus (-\infty, 1]$ and $\mathcal{D}_{-\frac{\pi}{2},\frac{\pi}{2}}(\pi)$.

After this transformation, our equation becomes

$$\frac{\mathrm{d}^2 W(\xi, u)}{\mathrm{d}\xi^2} = (u^2 + \psi(\xi))W(\xi, u)$$

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In this case, it is found that

$$W_2(\xi,u) = \mathrm{e}^{-u\xi}\left(1+\int_0^{+\infty}\mathrm{e}^{-ut}F_2(\xi,t)\,\mathrm{d}t\right) \sim \mathrm{e}^{-u\xi}\sum_{n=0}^{\infty}\frac{\mathsf{A}_n(\xi)}{u^n}$$

with

$$F_2(\xi,t) = \left(\mu^2-\tfrac{1}{4}\right)\frac{\mathrm{e}^{-\xi}}{2\sinh\xi}\mathbf{F}\left(\tfrac{3}{2}+\mu,\tfrac{3}{2}-\mu;2;\frac{\mathrm{e}^{-\xi}}{2\sinh\xi}(\mathrm{e}^{-t}-1)\right)\mathrm{e}^{-t}$$

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$$\mathsf{A}_n(\xi) = (-1)^n\sum_{k=0}^{n}\frac{S(n,k)}{k!}\left(\tfrac{1}{2}+\mu\right)_k\left(\tfrac{1}{2}-\mu\right)_k\left(\frac{\mathrm{e}^{-\xi}}{2\sinh\xi}\right)^k,$$

where $S(n,k)$ denote the Stirling numbers of the second kind. Note that the Borel summability holds in $\mathcal{D}_{-\frac{\pi}{2},\frac{\pi}{2}}(\pi)$, the theory says only $\mathcal{D}_{-\frac{\pi}{2},\frac{\pi}{2}}(\pi-\varepsilon)$. The (simple) singularities are at $t=-2\xi+2\pi\mathrm{i}n$, $n\in\mathbb{Z}$. Hence, Σ_ε can be enlarged to $\{t : -2\xi-2\pi\mathrm{i} < \mathfrak{Im}\,t < -2\xi+2\pi\mathrm{i}\}$.

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If $\mu+\nu \in \mathbb{C}\setminus\{-1,-2,\dots\}$, the standard subdominant solution is the associated Legendre function of the second kind Q_ν^μ. It is known that

$$\sqrt{\sinh\xi}\,Q_\nu^\mu(\cosh\xi) = \mathrm{e}^{\pi\mathrm{i}\mu}\sqrt{\frac{\pi}{2}}\,\Gamma\left(u+\mu+\tfrac12\right) \times \mathrm{e}^{-u\xi}\,\mathbf{F}\left(\tfrac12+\mu,\tfrac12-\mu;u+1;\frac{-\mathrm{e}^{-\xi}}{2\sinh\xi}\right),$$

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Resurgence in the high-order coefficients

In the general theorem on Borel summability, if $|\xi| < \min(\gamma - \varepsilon, \varepsilon)$ and $|\arg \xi| < \pi$ then $t = -2\xi$ is the closest singularity of the Borel transform $F_2(\xi, t)$ to the origin in Σ_ε. Thus, if δ is an arbitrarily small positive number, Darboux's theorem tells us that for $|\xi| < \frac{\gamma}{2} - \delta$ and $|\arg \xi| < \pi$, it holds that

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as $n \to +\infty$. Relations of these type are called resurgence relations and play an important role in exponential asymptotics. Such expansions for the coefficients of WKB solutions were first derived using non-rigorous methods by physicist Dingle.

In the case of our example, $t = -2\xi$ is the closest singularity, precisely when $\xi \in \mathcal{D}_{-\frac{\pi}{2},\frac{\pi}{2}}(\frac{\pi}{2})$. Numerical experiments confirm the asymptotics of the high-order coefficients in this region.

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Future research

We can study the more general equation

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$$W^{\pm}(\xi,u) \sim \exp\left(\pm u\xi \pm \tfrac{1}{2}\int^{\xi} \phi(t)\,\mathrm{d}t\right)\left(1 + \sum_{n=1}^{\infty} \frac{\mathrm{A}_n^{\pm}(\xi)}{u^n}\right),$$

as $u \to +\infty$ and ξ lies in appropriate domains. We could show the Borel summability of the WKB solutions, but analyticity in the t-plane followed only near the origin (except when $\psi(\xi,u) = \psi_0(\xi)$). Connection formulae have not yet been studied.

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Thank you for your attention!