Spectral synthesis on discrete Abelian groups

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Variety: translation invariant closed subspace of $C(G)$.

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Spectral synthesis does not hold on $\mathbb{R}^2$. Moreover, there is a variety $0 \neq V \subset C(\mathbb{R}^2)$ not containing any exponential.

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if $I \subset \mathcal{M}$ is an ideal then $I^\perp = \{f \in C(\mathbb{R}) \colon f * \mu = 0 \ (\mu \in I)\}$ is a variety.

Theorem (Ehrenpreis, Malgrange 1955)

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Krull's theorem *If J is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$ and $f \in \mathbb{C}[x_1, \ldots, x_n] \setminus J$, then there is a differential operator D s.t. $Dp(0, \ldots, 0) = 0$ for every $p \in J$ and $Df(0, \ldots, 0) \neq 0$.*

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Then $V_a = \{c_1 \cdot a + c_2 \colon c_1, c_2 \in \mathbb{C}\}$ is a variety, and every exponential polynomial in V_a is constant $(c \cdot 1)$.

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Proof: It is true on every finitely generated subgroup of G by Lefranc's theorem, so it is true on G. $\square$

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The only exponential contained in V_Q is the identically 1 function, and the polynomials contained in V_Q are the functions $f + c$ where f is additive and c is constant.

Consequently, Q is not in the closure of polynomials contained in V_Q.

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GP = set of generalized polynomials on G.

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f is an almost matrix function if, for every finite $E\subset G$, there is a finite dimensional subspace of $C(G)$ invariant under the subsemigroup generated by E and containing f.

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If $f \in C(G)$ is such that $f(x_1 \cdots x_n)$ is decomposable for some $n > 1$, then f is an almost matrix function.

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For every $f \in C(\mathbb{R}^p)$, if $f(x_1 + \ldots + x_n)$ is decomposable for some $n > 1$, then $f(x_1 + \ldots + x_n)$ is decomposable for every $n > 1$.

Theorem (ML 2018)

Let G be a commutative semigroup with unit. For every $f \in C(G)$ the following are equivalent.

(i) *There is an $n \geq 2$ such that $f(x_1 + \ldots + x_n)$ is decomposable.*

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$\exists f \in C(F_\omega)$ s.t. $f(x_1 + \ldots + x_n)$ is decomposable for some n, but $f(x_1 + x_2)$ is not decomposable.

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Corollary

Generalized spectral synthesis fails on G if $r_0(G) \geq 2^{\aleph_0}$. In particular, it fails on $\mathbb{R}$ (as a discrete group).

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Let $J = \langle x_1^2, x_1 - x_2, x_1 - x_3, \ldots \rangle$. Then $x_1 \notin J$. Then $c = (0, 0, \ldots)$ and $D = \sum_{k=1}^{\infty} \frac{\partial}{\partial x_k}$ do.

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$\kappa =?$ (CH)$\Longrightarrow \kappa = \aleph_1$. Is $\kappa > \aleph_1$ consistent with ZFC?

Lefranc's theorem is based on Krull's theorem: If J is an ideal in $\mathbb{C}[x_1, \ldots, x_n]$ and $f \in \mathbb{C}[x_1, \ldots, x_n] \setminus J$, then there is a differential operator D s.t. $Dp(0, \ldots, 0) = 0$ for every $p \in J$, and $Df(0, \ldots, 0) \neq 0$.

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Conjecture

$\kappa = \aleph_1$, *independently of the value of the continuum.*

Theorem (R. Katz, M. Krebs, A. Shaheen 2014)

If $f : \mathbb{R}^2 \to \mathbb{C}$ *is such that the sum of the values of* f *at the vertices of any unit square is zero, then* $f \equiv 0$.

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$1 + f(ay) + f(by) + f(ay)f(by) = (a + f(ay)(1 + f(by)) = 0$
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Thank you for your attention