Finite sums of ridge functions on convex subsets of $\mathbb{R}^n$

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$E \subset \mathbb{R}^n$, $\mathbf{x} = (x_1, \ldots, x_n) \in E$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$,
$\mathbf{a} \cdot \mathbf{x} = \sum_{j=1}^{n} a_j x_j$.
$\varphi : \Delta(\mathbf{a}) \to \mathbb{R}$, $\Delta(\mathbf{a}) := \{\mathbf{a} \cdot \mathbf{x} : \mathbf{x} \in E\}$.

$\varphi(\mathbf{a} \cdot \mathbf{x})$ – ridge function, $\mathbf{a}$ – direction.

John [1955] ("plane waves"), Logan and Shepp [1975], Friedman and Stuetzle [1981], Huber [1985], Donoho and Johnstone [1989], V.N. Temlyakov[1996], V.E. Maiorov[1999], R.A. Aliev, V.E. Ismailov[2015], A. Pinkus[1999-2015].

Let $\mathbf{a}^i$ be pairwise linearly independent, $\Delta_i := \Delta(\mathbf{a}^i)$.

$$f(\mathbf{x}) = \sum_{i=1}^{m} \varphi_i(\mathbf{a}^i \cdot \mathbf{x}) : \mathbb{R}^n \supset E \to \mathbb{R}. \tag{1}$$

$\varphi : \forall x, y \in \mathbb{R} \Rightarrow \varphi(x+y) = \varphi(x) + \varphi(y)$ - additive function

Example (m=3): $f(x,y) := \varphi(x) + \varphi(y) - \varphi(x+y) \equiv 0.$

$-\infty \leq \alpha < \beta \leq +\infty, |h| < \beta - \alpha, \forall x \in J_h := (\alpha, \beta) \cap (\alpha - h, \beta - h) :$
$\Delta_h \varphi(x) := \varphi(x+h) - \varphi(x).$

N.G.De Bruijn[1951]
$W(\alpha, \beta)$ has the difference property iff $\forall h(|h| < \beta - \alpha)$ & $(\Delta_h \varphi \in W(J_h)) \Rightarrow \varphi(x) = g(x) + h(x)$, where $g \in W(\alpha, \beta)$, h is additive on (α, β).

Lemma 1. Classes $D^k(\alpha, \beta), C^k(\alpha, \beta), C^\infty(\alpha, \beta), A(\alpha, \beta), P(\alpha, \beta)$ have the difference property.

Since now $W \in \{D^k, C^k, C^\infty, A, P\}$.

Theorem 1. *Let $E \subset \mathbb{R}^n$ be nonempty and open. Then the following conditions are equivalent:*
1) $\mathbf{a}^i$ are linearly independent;
2)for each f of the form (1): $f \in W(E) \Rightarrow \varphi_i \in W(\Delta_i)$ for $i = \overline{1, m}$.

A closed convex set $E \subset \mathbb{R}^n$ such that $int(E) \neq \emptyset$, is called a convex body. A hyperplane $\Gamma = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{c} \cdot \mathbf{x} = \gamma\}$ is called a support hyperplane for the set E at a point $\mathbf{y} \in E$ if $\forall \mathbf{x} \in E : \mathbf{c} \cdot \mathbf{x} \geq \mathbf{c} \cdot \mathbf{y} = \gamma$. The boundary $\partial(E)$ of a convex body E is said to be smooth if every point $y \in \partial(E)$ has a unique support hyperplane.

$-\infty \leq \alpha < \beta \leq +\infty$,
$(\alpha, \beta) \to B_{(\alpha,\beta)}$:
1) $B_{(\alpha,\beta)} \ni \varphi = g + h$, where $g \in C(\alpha, \beta)$, h is additive
$\Rightarrow h(x) = cx$.
2)$\forall h \in \mathbb{R}, |h| < \beta - \alpha : \varphi \in B_{(\alpha,\beta)} \Rightarrow \Delta_h \varphi \in B_{J_h}$.
3) $\forall (c, d) \subset (\alpha, \beta) : \varphi \in B_{(\alpha,\beta)} \Rightarrow \varphi \in B_{(c,d)}$.

Measurable or locally bounded functions belong to some B.
One-side locally bounded functions do not belong to any B.

Theorem 2. *Let $E \subset \mathbb{R}^n$ be convex and open, let $f \in W(E)$ be a function of the form (1), $\varphi_i \in B_{\Delta_i}$ for $i = \overline{1, m}$. Then $\varphi_i \in W(\Delta_i)$ for $i = \overline{1, m}$.*
Example:
$$f(\mathbf{x}) = \psi^2(x_1) - 3\psi^2(x_1 + x_2) + 3\psi^2(x_1 + 2x_2) - \psi^2(x_1 + 3x_2) \equiv 0$$

Theorem 3. *Let E be a convex body in $\mathbb{R}^n$, $E \neq \mathbb{R}^n$ and $m \geq 2$. Then the following conditions are equivalent:*

1) the boundary of E is smooth;

2) for arbitrary $\mathbf{a}^1, \ldots, \mathbf{a}^m$ and arbitrary functions $\varphi_i \in B_{int(\Delta_i)}$ $(i = 1, \ldots, m)$, the continuity of a function f of the form (1) on E implies the continuity of the functions φ_i on the sets Δ_i.

3) for arbitrary $\mathbf{a}^1, \ldots, \mathbf{a}^m$ and arbitrary functions $\varphi_i \in B_{int(\Delta_i)}$ $(i = 1, \ldots, m)$, the continuity of a function f of the form (1) on E implies the local boundedness of the functions φ_i on the sets Δ_i.

Theorem 4. *Let E be a convex body in $\mathbb{R}^n$, f be a continuous function of the form (1) on E, $\varphi_i \in B_{int(\Delta_i)}$ $(i = 1, \ldots, m)$, and let $t_0 \in \Delta_i$ be a boundary point for some i. Then $\varphi_i(t) = o(-\log|t - t_0|)$ as $t \to t_0$, $t \in \Delta_i$, and the obtained logarithmic estimate is sharp.*

Theorem 5. *Let* $\lambda(u) = o(-\log u)$ *as* $u \to 0+$, $n = 2$, $E = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \, x_1 \geq 0, x_2 \geq 0\}$. *Then there exists a continuous function* f *on* E *of the form*

$$f(\mathbf{x}) = \varphi(x_1 + x_2) - \varphi(x_1 + 2x_2)$$

such that $\lambda(u) = o(\varphi(u))$ *as* $u \to 0+$.

For an integrable function f *on a set* E *of finite positive measure, we define its mean value on* E*:*

$$f_E := \frac{1}{|E|} \int\limits_E f(x)dx.$$

Definition. *Let* $-\infty \le a < b \le +\infty$. *A locally integrable function* $f : [a, b] \to \mathbb{R}$ *is said to belong to the class* $VMO[a, b]$, *iff the equality*

$$\lim_{d-c\to 0} \frac{1}{d-c} \int\limits_{[c,d]} |f(x) - f_{[c,d]}|dx = 0,$$

with the limit taken over the intervals
$[c, d] \subset [a, b], -\infty < c < d < +\infty.$

Theorem 6. *Let* E *be a convex body in* $\mathbb{R}^n$, f *be a continuous function of the form (1) on* E, $\varphi_i \in B_{int(\Delta_i)}$ $(i = 1, \dots, m)$, *and let* $[a, b] \subset \Delta_i, -\infty < a < b < +\infty$. *Then* $\varphi_i \in VMO[a, b]$.

For a function $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^n$ we define its modulus of continuity at a point $\mathbf{x}^0 \in E$:

$$\omega(f, \mathbf{x}^0, t) := \sup_{||\mathbf{h}|| \le t, \mathbf{x}^0 + \mathbf{h} \in E} |f(\mathbf{x}^0 + \mathbf{h}) - f(\mathbf{x}^0)|.$$

$$f \in C(E) \Rightarrow \omega(f, \mathbf{x}^0, \cdot) \nearrow \in C[0, +\infty], \omega(f, \mathbf{x}^0, 0) = 0.$$

Theorem 7. *Let E be a convex body in $\mathbb{R}^n$, f be a continuous function of the form (1) on E, $\varphi_i \in B_{int(\Delta_i)}$ $(i = 1, \dots, m)$. Let a be a boundary point of the set Δ_i and for some $\mathbf{x}^0 \in E$ such that $\mathbf{a}^i \cdot \mathbf{x}^0 = a$, the modulus of continuity $\omega(t) := \omega(f, \mathbf{x}^0, t)$ satisfies the Dini condition*

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty. \qquad (2)$$

Then there exists a finite limit

$$A := \lim_{t \to a} \varphi_i(t).$$

Theorem 8. *Let* $\eta : [0, +\infty) \to \mathbb{R}$ *be an arbitrary continuous nondecreasing function with* $\eta(0) = 0$, *which does not satisfy the condition (2) .Let* $n = 2$,
$E = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \ x_1 \geq 0, x_1 \leq x_2 \leq 2x_1\}$. *Then there exists a continuous function* f *on* E *of the form*

$$f(\mathbf{x}) = \varphi(x_2) - \varphi(x_1)$$

such that

$$\omega(f, \mathbf{0}, \delta) = \eta(\delta)$$

for $\delta \geq 0$, $\varphi \in C(0, +\infty)$, $\varphi(0) = 0$, $\lim\limits_{t \to +0} \varphi(t) = -\infty$.

Theorem 9. *Let* E *be a convex body in* $\mathbb{R}^n$, f *be a continuous function of the form (1) on* E,$\varphi_i \in B_{int(\Delta_i)}$ $(i = 1, \ldots, m)$. *Suppose that for some* $\mathbf{x}^0 \in E$ *and for all* $i = 1, \ldots, m$ *there exist finite limits* $\lim\limits_{t \to a_i} \varphi_i(t) =: b_i$, *where*
$a_i := \mathbf{a}^i \cdot \mathbf{x}^0$. *Then the simultaneous replacement of all the values* $\varphi_i(a_i)$ *by the numbers* b_i $(i = 1, \ldots, m)$ *does not change the values of the function* f *on* E.

Theorem 10. *Let E be a convex body in $\mathbb{R}^n$, $E \neq \mathbb{R}^n$ and $k \geq 0$. Then the following conditions are equivalent:*

1) the boundary of E is smooth;

2) for arbitrary $m \in \mathbb{N}$, for arbitrary vectors $\mathbf{a}^1, \dots, \mathbf{a}^m$ and arbitrary functions $\varphi_i \in B_{int(\Delta_i)}$ $(i = 1, \dots, m)$, we have $f \in C^k(E) \Rightarrow \varphi_i \in C^k(\Delta_i)$ $(i = 1, \dots, m)$, where f is of the form (1).

Publications

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Publications

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Thanks for your attention!