Density of weighted multivariate polynomials

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1. Weierstrass Theorem and its variations

Trigonometric Weierstrass approximation theorem: any 2π periodic continuous function $f(x)$ is a uniform limit on $[-\pi, \pi]$ of trigonometric polynomials of degree n as $n \to \infty$. Substituting $t = \tan \frac{x}{2}, x \in (-\pi, \pi)$ transforms 2π periodic continuous functions into continuous functions $f \in C_0(\mathbb{R})$ which have equal finite limits at $\pm\infty$, and the trigonometric polynomials of degree n become rational functions $(1 + t^2)^{-n} p_{2n}(t)$ with $p_{2n}(t)$ being an algebraic polynomial of degree at most $2n$.

This leads to an equivalent version of the trigonometric Weierstrass theorem:

Every $f \in C_0(\mathbb{R})$ with equal finite limits at $\pm\infty$ is a uniform limit on $\mathbb{R}$ of weighted algebraic polynomials

$$w(t)^{-2n} p_{2n}(t), \quad w(t) := \sqrt{1 + t^2}, \quad \deg p_{2n} \le 2n.$$

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Problem: *Characterize those weights for which density in $C_0(\mathbb{R}^d)$ holds with weighted polynomials*

$$w^{-n} p_n, \quad \deg p_n \leq n.$$

Clearly, we must have $w(\mathbf{x}) \geq c|\mathbf{x}|$ in order for $w^{-n} p_n$ to be bounded in $\mathbb{R}^d$.

Above problem received a considerable attention in case when the even weight $w(t)$ grows at ∞ *faster than* t. Obviously this implies that

$$w(t)^{-n} p_n(t) \to 0, t \to \pm\infty$$

for all polynomials of degree at most n that is weighted polynomials can not provide uniform approximation on all of the real line. In this case weighted polynomials can be dense only for functions with *finite support* and this finite domain of approximation which depends on w is determined by methods of potential theory.

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A model case of this phenomena is provided by Freud weights $w_\alpha(t) := e^{|t|^\alpha}$. If $\alpha \geq 1$ a function $f \in C(\mathbb{R})$ is a uniform limit of $w_\alpha(t)^{-n}p_n(t)$, $\deg p_n \leq n$, if and only if f vanishes outside the interval $[-a_\alpha, a_\alpha]$ with a_α being a certain parameter depending only on α.

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A similar situation occurs in case of approximation by *incomplete* polynomials given by $\sum_{n\theta \leq k \leq n} a_k x^k, 0 < \theta < 1$ which can be regarded as weighted polynomials $x^{n\theta/(\theta-1)}p_n$, $\deg p_n \leq n$, i.e., $w_\theta(x) := x^{\theta/(1-\theta)}$ in this case. Then $f \in C[0,1]$ can be uniformly approximated by a sequence of θ-incomplete polynomials if and only if it vanishes on $[0, \theta^2]$. Hence similarly to the case of Freud weights the functions must vanish on a substantial part of their domain in order for the weighted approximation to hold.

Consider the space $C_0(\mathbb{R}^d)$ of continuous functions with equal limits at infinity along lines passing through the origin , i.e.,

$$C_0(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \exists r_f \in C(S^{d-1}),\ \lim_{|t|\to\infty} f(t\mathbf{x}) = r_f(\mathbf{x}),\ \mathbf{x} \in S^{d-1}\}.$$

Given a positive even weight w on $\mathbb{R}^d$ we approximate $f \in C_0(\mathbb{R}^d)$ by weighted polynomials $w^{-n}p_n$ on $\mathbb{R}^d$, where $p_n \in P_n^d$ are multivariate polynomials of d variables of degree at most n. Assume in addition, that $tw(\frac{\mathbf{x}}{t})$ is monotone increasing for $t > 0$ for every fixed $\mathbf{x} \in \mathbb{R}^d$, and has a continuous positive limit as $t \to 0$. Then, in particular,

$$w(t\mathbf{x}) \sim |t|w(\mathbf{x}), t \to \infty$$

i.e. the weight is of order $|t|$ at infinity. Such weights will be called **admissible**. Note that for admissible weights $w^{-2n}p_{2n} \in C_0(\mathbb{R}^d)$ for any $p_{2n} \in P_{2n}^d$ and $n \in \mathbb{N}$.

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Given an admissible weight w we homogenize it setting

$$w^*(\mathbf{x}, t) := |t|w\left(\frac{\mathbf{x}}{t}\right) : \mathbb{R}^{d+1} \mapsto \mathbb{R}^+,\ \mathbf{x} \in \mathbb{R}^d,\ t \neq 0.$$

Example. $w(\mathbf{x}) = \sqrt{1 + x_1^2 + ... + x_d^2}$, $w^*(\mathbf{x}, t) = \sqrt{t^2 + x_1^2 + ... + x_d^2}$.

Theorem. (AK, 2019) *Let w be a convex admissible weight on $\mathbb{R}^d, d \geq 1$. In addition, if $d > 1$ assume that w is piecewise C^1, i.e., with some $s \in \mathbb{N}$ we have $w = \max\{w_j : 1 \leq j \leq s\}$ where each w_j is admissible convex and $w_j^* \in C^1(\mathbb{R}^{d+1} \setminus \{0\}), 1 \leq j \leq s$. Then for every $f \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that*

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Thus when $d = 1$ the **convexity** of the admissible weight yields the density of weighted polynomials $w^{-2n} p_{2n}$ in the space $C_0(\mathbb{R})$. If $d > 1$ we need in addition the piecewise C^1 smoothness of weights in order for the density to hold. It is plausible that the convexity of admissible weights should suffice for the density of weighted polynomials $w^{-2n} p_{2n}$ in $C_0(\mathbb{R}^d)$ when $d > 1$, as well. Thus we would like to offer the next conjecture which would provide a full analogue of weighted Weierstrass approximation theorem in $\mathbb{R}^d$.

Conjecture. *For any convex admissible weight on $\mathbb{R}^d, d \geq 1$ and $f \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that*

$$w^{-2n} p_{2n} \to f,\ n \to \infty$$

uniformly on $\mathbb{R}^d$.

Example. Let l_α norm of $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$ be given by the relations

$$|\mathbf{x}|_\alpha := \begin{cases} (|x_1|^\alpha + ... + |x_d|^\alpha)^{\frac{1}{\alpha}}, \\ \max_{1\le j\le d} |x_j|, \ \alpha = \infty. \end{cases}$$

Consider the admissible weights

$$w_\alpha(\mathbf{x}) := (1 + |x_1|^\alpha + ... + |x_d|^\alpha)^{\frac{1}{\alpha}} = (1 + |\mathbf{x}|_\alpha^\alpha)^{\frac{1}{\alpha}}, \ \mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d.$$

Note that w_α is **convex** on $\mathbb{R}^d$ if $\alpha \ge 1$. It is also easy to check that these weights are C^1 for $1 < \alpha < \infty$ and piecewise C^1 if $\alpha = 1, \infty$. Weights $w_\alpha(\mathbf{x}), \alpha \ge 1$ provide a model of weights for which conditions of Theorem 1 hold.

Corollary. *Let $1 \le \alpha \le \infty$. Then for every $f \in C_0(\mathbb{R}^d), d \ge 1$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that*

$$w_\alpha^{-2n} p_{2n} \to f, \ n \to \infty$$

uniformly on $\mathbb{R}^d$.

2. Weierstrass type weighted approximation with non convex weights

Admissibility of the weight appears to be a natural requirement for approximating every function in $C_0(\mathbb{R}^d)$. How about the convexity of the weight? For instance, $w_\alpha(\mathbf{x}) = (1 + |x_1|^\alpha + ... + |x_d|^\alpha)^{\frac{1}{\alpha}}$ when $0 < \alpha < 1$?

It turns out that non convexity of the weight changes the situation drastically. Indeed, it was proved recently by Kroó and Totik that

When $d = 1$ and $0 < \alpha < 1$ there exist weighted polynomials $w_\alpha^{-2n} p_{2n}, p_n \in P_{2n}^1$, $n \in \mathbb{N}$, converging to $f \in C(\mathbb{R})$ uniformly on $\mathbb{R}$ if and only if $f(0) = f(\infty) = f(-\infty) = 0$.

Here $f(\infty), f(-\infty)$ stand for the corresponding limits at infinity. Thus some additional restrictions need to be imposed on the function, namely it must vanish at a certain *exceptional set*.

Now consider the multivariate approximation by $w_\alpha^{-2n} p_{2n}, p_{2n} \in P_{2n}^d$ when $\alpha < 1$, that is the weight is not convex.

What are the exceptional sets in the multivariate case?

Denote by

$$L^d := \{\mathbf{x} = (x_1, ..., x_d) \in K_\alpha^d : x_1 \cdot ... \cdot x_d = 0\}$$

the union of all coordinate planes.

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Now we identify the exceptional zero set for functions admitting weighted polynomial approximation on $\mathbb{R}^d, d > 1$ with the non convex $w_\alpha, \alpha < 1$. Essentially, in multivariate case the exceptional zero set consists of the

union of all coordinate planes L^d and the infinity.

Theorem. (AK, 2019) *Let* $0 < \alpha < 1$ *and* $d \geq 2$. *If* $f \in C_0(\mathbb{R}^d)$ *is a uniform limit on* $\mathbb{R}^d$ *of weighted polynomials* $w_\alpha^{-2n} p_{2n}, p_{2n} \in P_{2n}^d$ *then necessarily* $f = 0$ *on* $L^d \cup \{\infty\}$. *Moreover, if* $0 < \alpha < 1$ *is rational then any* $f \in C(\mathbb{R}^d)$ *which vanishes on* $L^d \cup \{\infty\}$ *is a uniform limit on* $\mathbb{R}^d$ *of weighted polynomials* $w_\alpha^{-2n} p_{2n}, p_{2n} \in P_{2n}^d$.

The sufficiency in the above theorem for irrational $0 < \alpha < 1$ (and $d > 1$) is an open problem.

3.Density of homogeneous polynomials on 0-symmetric star like domains

The problem of approximating $f \in C_0(\mathbb{R}^d)$ by weighted polynomials $w^{-2n}p_{2n}$ uniformly on $\mathbb{R}^d$ is closely related to uniform approximation on the boundary of *0-symmetric star like domains* by multivariate **homogeneous** polynomials

$$h \in H_n^d := \{\sum_{|\mathbf{k}|=n} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R}\}.$$

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In fact a certain duality can be established between this two problems.

An admissible weight $w \in C(\mathbb{R}^d)$ is associated with the 0-symmetric star like domain in $\mathbb{R}^{d+1}$

$$K_w := \{(\mathbf{x}, t) \in \mathbb{R}^{d+1} : w^*(\mathbf{x}, t) \leq 1\},$$

where $w^*(\mathbf{x}, t)$ is the homogenization of w. K_w is convex whenever w is convex.

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Conversely, when K is a 0-symmetric star like set of points $\mathbf{z} = (\mathbf{x}, t) \in \mathbb{R}^{d+1}$ with the Minkowski functional $\phi_K(\mathbf{z}) := \inf\{\alpha > 0 : \frac{\mathbf{z}}{\alpha} \in K\}$ we can associate this set with an even positive weight on $\mathbb{R}^d$ defined by the relation

$$w_K(\mathbf{x}) := \phi_K(\mathbf{x}, 1), \quad \mathbf{x} \in \mathbb{R}^d.$$

The next statement gives a duality between the problem of approximating $f \in C_0(\mathbb{R}^d)$ by weighted polynomials $w^{-2n}p_{2n}$ uniformly on $\mathbb{R}^d$ and uniform approximation on the boundary of *0-symmetric star like domains* by multivariate **homogeneous** polynomials.

Duality Principle. (i) Let $w \in C(\mathbb{R}^d), d \geq 1$ be an admissible weight on $\mathbb{R}^d$. If for $\forall g \in C_0(\mathbb{R}^d)$ there exist $p_{2n} \in P_{2n}^d$ so that $w^{-2n}p_{2n} \to g,\ n \to \infty$ uniformly on $\mathbb{R}^d$ then for each even $f \in C(\partial K_w)$ there exist homogeneous polynomials $h_{2n} \in H_{2n}^{d+1}$ for which $f = \lim h_{2n}$ uniformly on ∂K_w.

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(ii) Conversely, let K be any 0-symmetric star like set of points $(\mathbf{x}, t) \in \mathbb{R}^{d+1}$. Assume that for each even function $f \in C(\partial K)$ there exist homogeneous polynomials $h_{2n} \in H_{2n}^{d+1}$ such that $f = \lim h_{2n}$ uniformly on ∂K. Then for every $g \in C_0(\mathbb{R}^d)$ there exist polynomials $p_{2n} \in P_{2n}^d$ so that $w_K^{-2n}p_{2n} \to g,\ n \to \infty$ uniformly on $\mathbb{R}^d$.

The following conjecture which may be regarded as Weierstrass type density theorem for homogeneous polynomials attracted considerable attention in the last decade

Conjecture: *For any 0-symmetric convex body* $K \subset \mathbb{R}^d$ *and every even* $f \in C(\partial K)$ *there exist homogeneous polynomials* $h_{2n} \in H_{2n}^d$ *such that* $f = \lim_{n\to\infty} h_{2n}$ *uniformly on* ∂K.

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The conjecture has been verified in the following 3 cases:

(i) When **d=2** (Benko-Kroó, Varju)

(ii) For any **0**-symmetric convex **polytope** in $\mathbf{R}^d, d > 2.$ (Varju)

(iii) For any **0**-symmetric **regular** convex body $K \subset \mathbf{R}^d, d > 2.$ (Kroó-Szabados)

Note: In case of arbitrary $f \in C(\partial K)$ above statements hold for $h_{2n} + h_{2n+1} \in H_{2n}^d + H_{2n+1}^d$ so that $f = \lim_{n\to\infty}(h_{2n} + h_{2n+1})$ uniformly on ∂K.

What happens with homogeneous polynomial approximation on non convex 0-symmetric convex bodies $K \subset \mathbb{R}^d$?

It turns out that for every 0-symmetric star like domain K in $\mathbb{R}^d$ there exists an exceptional 0-symmetric set $Z(K) \subset \partial K$ so that for any even $f \in C(\partial K)$ the following statements are equivalent

(i) there exist $h_{2n} \in H_{2n}^d$ such that $f = \lim_{n\to\infty} h_{2n}$ uniformly on ∂K

(ii) $f = 0$ on $Z(K)$.

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(ii) $f = 0$ on $Z(K)$.

Example. Consider the l_α ball

$$B_\alpha := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}|_\alpha = |x_1|^\alpha + ... + |x_d|^\alpha \leq 1\}.$$

When $0 < \alpha < 1$ this set is not convex. In this case the exceptional zero set is the intersection of ∂B_α with the union of coordinate planes

$$Z(B_\alpha) = \{\mathbf{x} = (x_1, ..., x_d) : |x_1|^\alpha + ... + |x_d|^\alpha = 1,\ x_1 \cdot ... \cdot x_d = 0\}.$$

Hence uniform approximation by homogeneous polynomials on $\partial(B_\alpha), 0 < \alpha < 1$ is possible if and only if f vanishes on $Z(B_\alpha)$ (Kroó-Totik (2018), $d = 2$, Kroó(2019), $d > 2, \alpha$ and is rational).

4.Density of multivariate weighted polynomials $w^{\gamma_n} p_n$

In case of approximation by weighted polynomials $w^n p_n$, $\deg p_n \le n$, we either can ensure density only on part of the domain (Freud weights, incomplete polynomials), or we need to assume the growth condition $w(x) \sim \frac{1}{|x|}$ at infinity. But what happens if w^n is replaced by w^{γ_n} with $\gamma_n = o(n)$? Then this growth restriction is not needed, in general.

Question: Are weighted polynomials $w^{\gamma_n} p_n$, $\deg p_n \le n$ dense in $C(K)$?

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Question: Are weighted polynomials $w^{\gamma_n}p_n$, $\deg p_n \le n$ dense in $C(K)$?

Given a a closed subset $K \subset \mathbb{R}^d$ and a nonnegative weight $w \in C(K)$ we want to approximate $f \in C(K)$ by weighted polynomials $w^{\gamma_n}p_n, p_n \in P_n^d, \gamma_n = o(n)$. When $\infty \in K$, in order for the inclusion $w^{\gamma_n}P_n^d \subset C(K)$ to hold for each n, we will need that $|\mathbf{x}|^k w(\mathbf{x}) \to 0, |\mathbf{x}| \to \infty, \forall k > 0$. This problem is nontrivial only if the zero set of the weight $Z_w := \{\mathbf{x} \in K : w(\mathbf{x}) = 0\}$ is not empty. But any uniform limit of $w^{\gamma_n}p_n$ must vanish on Z_w, i.e., we always have the inclusion $\lim_{n\to\infty} w^{\gamma_n}P_n^d \subset \{f \in C(K) : f = 0 \text{ on } Z_w\}$.

When do we have the equality

$$\lim_{n\to\infty} w^{\gamma_n}P_n^d = \{f \in C(K) : f = 0 \text{ on } Z_w\}?$$

First we answer above question for weights with *polynomial singularities* and *bounded* sets K. Let us denote by $J[a,b]$ the class of all weights w on $[a,b]$ with finite polynomial singularities which can be written in the form

$$w(x) = w_0(x) \prod_{1 \le j \le s} |x - a_j|^{\alpha_j}$$

with some $s \in \mathbf{N}$, $a \le a_1 < \cdots < a_s \le b, \alpha_j > 0$, and w_0 which is positive and analytic in an open neighborhood of $[a,b]$. Note that $s \in \mathbf{N}$, analytic functions w_0, singularities a_j and their multiplicities α_j may vary for different $w \in J[a,b]$. For any $q \in C(K)$ we denote by $\Omega_q := [\min_{\mathbf{x} \in K} q(\mathbf{x}), \max_{\mathbf{x} \in K} q(\mathbf{x})]$ the range of q.

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Theorem. (A.Kroó, J. Szabados, 2019) *Let* $K \subset \mathbb{R}^d, d \ge 1$ *be a closed bounded set. For any polynomial* $q \in P^d$ *and any weight* $w^* \in J(\Omega_q)$ *consider the multivariate weight* $w(\mathbf{x}) := w^*(q(\mathbf{x})), \mathbf{x} \in \mathbb{R}^d$. *Then*

$$\lim_{n \to \infty} w^{\gamma_n} P_n^d = \{ f \in C(K) : f = 0 \text{ on } Z_w \}$$

if and only if $\gamma_n = o(n)$.

Thus when $\gamma_n = o(n)$, approximation by weighted polynomials holds for a wide class of multivariate Jacobi type weights on all of the underlying domain with the necessary exception of the zero set of the weight.

A similar result can be verified for approximation by Freud type weights when the n-th power of the weight is replaced by $\gamma_n = o(n)$. Given a multivariate polynomial $q \in P^d, d \geq 1$ and $\alpha \geq 1$ consider weighted polynomials of the form

$$e^{-\gamma_n |q(\mathbf{x})|^\alpha} p_n(\mathbf{x}), \quad p_n \in P_n^d.$$

Naturally, when $d \geq 2$ in order for $e^{-\gamma_n |q(\mathbf{x})|^\alpha} p_n(\mathbf{x})$ to be bounded on $\mathbb{R}^d$ we need to assume that $\frac{|q(\mathbf{x})|}{\log |\mathbf{x}|} \to \infty, \quad |\mathbf{x}| \to \infty$. In fact under this assumption the weighted polynomials tend to zero at infinity and hence only $f \in C(\mathbb{R}^d), f(\infty) = 0$ can be approximated.

A similar result can be verified for approximation by Freud type weights when the n-th power of the weight is replaced by $\gamma_n = o(n)$. Given a multivariate polynomial $q \in P^d, d \geq 1$ and $\alpha \geq 1$ consider weighted polynomials of the form

$$e^{-\gamma_n |q(\mathbf{x})|^\alpha} p_n(\mathbf{x}), \quad p_n \in P_n^d.$$

Naturally, when $d \geq 2$ in order for $e^{-\gamma_n |q(\mathbf{x})|^\alpha} p_n(\mathbf{x})$ to be bounded on $\mathbb{R}^d$ we need to assume that $\frac{|q(\mathbf{x})|}{\log |\mathbf{x}|} \to \infty, \quad |\mathbf{x}| \to \infty$. In fact under this assumption the weighted polynomials tend to zero at infinity and hence only $f \in C(\mathbb{R}^d), f(\infty) = 0$ can be approximated.

Theorem. (A.Kroó, J. Szabados, 2019) *Let $\gamma_n \uparrow \infty$ be a sequence of real numbers increasing to infinity, and $\alpha \geq 1$. Then for any polynomial q as above we have*

$$\lim_{n\to\infty} e^{-\gamma_n |q(\mathbf{x})|^\alpha} P_n^d = \{f \in C(\mathbb{R}^d), f(\infty) = 0\}$$

if and only if $\gamma_n = o(n)$.

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