Simultaneous observability of infinitely many strings and beams

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Abstract

We report on a joint work with A. C. Lai and P. Loreti.

We investigate the simultaneous observability of infinite systems of vibrating strings or beams having a common endpoint where the observation is taking place.

Our results are new even for finite systems because we allow the vibrations to take place in independent directions.

Our main tool is a vectorial generalization of some classical theorems of Ingham and Beurling in nonharmonic analysis.

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A system of strings in the space

We consider a system of strings in $\mathbb{R}^3$. The jth string has fixed endpoints in 0 and $(\ell_j, \varphi_j, \theta_j)$ in spherical coordinates, and it has a transversal vibration in a plane determined by a unit vector $v_j \perp (\ell_j, \varphi_j, \theta_j)$. Denoting by $u_j(t, r, \varphi_j, \theta_j)v_j$ the transversal displacement of the j'th string at time $t \in \mathbb{R}$ at the point (r, φ_j, θ_j), we consider the following uncoupled system for $j = 1, \ldots, J$ (we omit the arguments φ_j, θ_j and the subscripts t and r denote differentiations):

$$\begin{cases} u_{j,tt} - u_{j,rr} = 0 & \text{in} \quad \mathbb{R} \times (0, \ell_j), \\ u_j(t,0) = u_j(t,\ell_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_j(0,r) = u_{j0}(r) & \text{for} \quad r \in (0, \ell_j), \\ u_{j,t}(0,r) = u_{j1}(r) & \text{for} \quad r \in (0, \ell_j). \end{cases}$$

Simultaneous observability

Theorem

Assume that

$$\ell_j/\ell_m \quad \textit{is irrational for all} \quad j \neq m.$$

There exists a number

$$T_0 \leq 2(\ell_1 + \cdots + \ell_J)$$

such that the restricted linear map

$$(u_{10}, u_{11}, \ldots, u_{J0}, u_{J1}) \mapsto \sum_{j=1}^{J} u_{j,r}(\cdot, 0) v_j|_I$$

is one-to-one for every interval I of length $> T_0$, and for no interval I of length $< T_0$.

Remarks on the critical time

We have

$$T_0 \geq 2\max\{\ell_1, \ldots, \ell_J, (\ell_1 + \cdots + \ell_J)/3\}.$$

Examples

(i) If $J = 3$ and v_1, v_2, v_3 are mutually orthogonal, then

$$T_0 = 2\max\{\ell_1, \ell_2, \ell_3\}.$$

(ii) If $J = 3$ and $v_1 \perp v_2 = v_3$, then

$$T_0 = 2\max\{\ell_1, \ell_2 + \ell_3\}.$$

(iii) If all vectors v_j are equal ("planar case"), then

$$T_0 = 2(\ell_1 + \cdots + \ell_J).$$

An infinite system of loaded strings

The theorem may be proved either by using d'Alembert's formula or Fourier series representations.

The method of Fourier series enables us to study more general equations, and even infinite systems of strings.

Thus we consider the following system with $j = 1, 2, \ldots,$ with arbitrarily given numbers $a_j \ge 0$:

$$\begin{cases} u_{j,tt} - u_{j,rr} + a_j u_j = 0 & \text{in} \quad \mathbb{R} \times (0, \ell_j), \\ u_j(t,0) = u_j(t,\ell_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_j(0,r) = u_{j0}(r) & \text{for} \quad r \in (0,\ell_j), \\ u_{j,t}(0,r) = u_{j1}(r) & \text{for} \quad r \in (0,\ell_j). \end{cases}$$

Well posedness and hidden regularity

We consider the closed subspace H of $\prod_{j=1}^{\infty}(H_0^1(0,\ell_j)\times L^2(0,\ell_j))$ defined by

$$\|(u_{10},u_{11},u_{20},u_{21},\ldots)\|_H^2 := \sum_{j=1}^{\infty}\frac{1}{\ell_j}\left(\|u_{j0}\|_{H_0^1(0,\ell_j)}^2+\|u_{j1}\|_{L^2(0,\ell_j)}^2\right)<\infty.$$

Proposition

The linear map

$$(u_{10},u_{11},u_{20},u_{21},\ldots)\mapsto\sum_{j=1}^{\infty}u_{j,r}(\cdot,0)v_j$$

is well defined and continuous from H into $L^2_{loc}(\mathbb{R};\mathbb{R}^3)$.

Simultaneous observability

Set $\omega_{j,k} := \sqrt{\left(\frac{k\pi}{\ell_j}\right)^2 + a_j}$, and assume that

$$(j_1, k_1) \ne (j_2, k_2) \Longrightarrow \omega_{j_1,k_1} \ne \omega_{j_2,k_2}.$$

Theorem

There exists a number

$$T_0 \in \left[2 \max \left\{ \frac{\sum \ell_j}{3}, \ell_1, \ell_2, \ldots \right\}, 2 \sum \ell_j \right]$$

such that the map

$$(u_{10}, u_{11}, u_{20}, u_{21}, \ldots) \mapsto \sum_{j=1}^{\infty} u_{j,r}(\cdot, 0) v_j|_I$$

is one-to-one if $|I| > T_0$*, and not one-to-one if* $< T_0$*.*

Beam systems

Next we consider a system of transversally vibrating hinged beams with unit vectors $v_j \perp (\ell_j, \varphi_j, \theta_j)$ as before, so that

$$\begin{cases} u_{j,tt} + u_{j,rrrr} = 0 & \text{in} \quad \mathbb{R} \times (0, \ell_j), \\ u_j(t,0) = u_j(t,\ell_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_{j,rr}(t,0) = u_{j,rr}(t,\ell_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_j(0,r) = u_{j0}(r) & \text{for} \quad r \in (0, \ell_j), \\ u_{j,t}(0,r) = u_{j1}(r) & \text{for} \quad r \in (0, \ell_j), \\ j = 1, 2, \ldots . \end{cases}$$

Well posedness and hidden regularity

We consider the closed subspace H of $\prod_{j=1}^{\infty}(H_0^1(0,\ell_j)\times H^{-1}(0,\ell_j))$ defined by

$$\|(u_{10},u_{11},u_{20},u_{21},\ldots)\|_H^2 := \sum_{j=1}^{\infty}\frac{1}{\ell_j}\left(\|u_{j0}\|^2_{H_0^1(0,\ell_j)}+\|u_{j1}\|^2_{H^{-1}(0,\ell_j)}\right)<\infty.$$

Proposition

The linear map

$$(u_{10},u_{11},u_{20},u_{21},\ldots)\mapsto\sum_{j=1}^{\infty}u_{j,r}(\cdot,0)v_j$$

is well defined and continuous from H into $L^2_{loc}(\mathbb{R};\mathbb{R}^3)$.

An observability theorem for beam systems

Theorem

Assume that

$$(\ell_j/\ell_m)^2 \quad \textit{is irrational for all} \quad j \neq m.$$

If I is an interval of length

$$|I| > T_0 := 2\pi \limsup_{r\to\infty} \frac{1}{r} \sum_{j=1}^{\infty} \left[\frac{\sqrt{r}\ell_j}{\pi} \right],$$

then the restricted linear map

$$(u_{10}, u_{11}, u_{20}, u_{21}, \ldots) \mapsto \sum_{j=1}^{\infty} u_{j,r}(\cdot, 0) v_j|_I$$

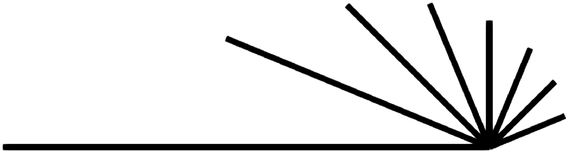
is one-to-one.

We may have $T_0 < \infty$ even if $\sum \ell_j = \infty$

If

$$\ell_j = \frac{\pi}{\sqrt{j+\sqrt{2}}}, \quad j = 1, 2, \dots,$$

then $7 < T_0 < 11$:



More generally, we have $T_0 < \infty$ if

$$\sum_{\ell_j \ge x} \ell_j = O\left(\frac{1}{x}\right) \quad \text{as} \quad x \to 0.$$

Upper density of a sequence

Every increasing sequence $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ has an upper density

$$D^+ = D^+(\{\omega_k \ : \ k \in \mathbb{Z}\}) := \lim_{r\to\infty} \frac{n^+(r)}{r} \in [0, \infty]$$

where

$$n^+(r) := \sup_{x\in\mathbb{R}} |\{\omega_k\} \cap [x, x + r]| \,.$$

Examples

- If $\gamma := \inf_{k\in\mathbb{Z}} (\omega_{k+1} - \omega_k) > 0$ (*uniform separation*), then $D^+ \le \frac{1}{\gamma}$.
- If $\omega_k = k$ for all $k \in \mathbb{Z}$, then $D^+ = \gamma = 1$.
- If $\omega_k = k\,|k|$ for all $k \in \mathbb{Z}$, then $\gamma = 1$ and $D^+ = 0$.

A vectorial Ingham–Beurling type theorem

Let $(\omega_k)_{k\in\mathbb{Z}}$ be a uniformly separated, increasing sequence of real numbers, and $(U_k)_{k\in\mathbb{Z}}$ a sequence of unit vectors in some complex Hilbert space G.

Theorem

We have

$$c_1(I)\sum_{k\in\mathbb{Z}}|x_k|^2 \le \int_I \Big\|\sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}\Big\|_G^2 \, dt \le c_2(I)\sum_{k\in\mathbb{Z}}|x_k|^2$$

for every bounded interval I of length $|I| > 2\pi D^+$.

An extension to not uniformly separated sequences

Let $(\omega_k)_{k\in\mathbb{Z}}$ be an increasing sequence of real numbers, having a finite upper density D^+, and $(U_k)_{k\in\mathbb{Z}}$ a sequence of unit vectors in some complex Hilbert space G.

Theorem

1. *We have*

$$\int_I \Big\| \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t} \Big\|_G^2 \, dt \le c_2(I) \sum_{k\in\mathbb{Z}} |x_k|^2$$

for every bounded interval I.

2. *If*

$$t \mapsto \sum_{k\in\mathbb{Z}} x_k U_k e^{i\omega_k t}$$

vanishes in an interval I of length $> 2\pi D^+$, then all coefficients x_k vanish.

Explanation of the second part

For the second part we may reduce the problem to the scalar case by expressing every vector U_k in some fixed orthonormal basis of G. Furthermore, by density it suffices to consider finite sums.

By a theorem of Baiocchi–K.–Loreti there exists a (linear algebraic) basis $(f_k(t))$ of the linear span of the functions $(e^{i\omega_k t})$ such that if we rewrite our sums in this basis:

$$\sum_{k\in\mathbb{Z}} x_k e^{i\omega_k t} = \sum_{k\in\mathbb{Z}} y_k f_k(t),$$

then

$$y_k = 0 \quad \text{for all} \quad k \Longleftrightarrow x_k = 0 \quad \text{for all} \quad k,$$

and

$$\sum_{k\in\mathbb{Z}} |y_k|^2 \le c(I) \int_I \Big| \sum_{k\in\mathbb{Z}} x_k e^{i\omega_k t} \Big|^2 \, dt$$

whenever $|I| > 2\pi D^+$.

Observability of the string system

We recall the system that we are studying:

$$\begin{cases} u_{j,tt} - u_{j,rr} + a_j u_j = 0 & \text{in} \quad \mathbb{R} \times (0, \ell_j), \\ u_j(t,0) = u_j(t,\ell_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_j(0,r) = u_{j0}(r) & \text{for} \quad r \in (0,\ell_j), \\ u_{j,t}(0,r) = u_{j1}(r) & \text{for} \quad r \in (0,\ell_j), \\ j = 1, 2, \ldots. \end{cases}$$

We are going to show that the linear map

$$(u_{10}, u_{11}, u_{20}, u_{21}, \ldots) \mapsto \sum_{j=1}^{\infty} u_{j,r}(\cdot, 0) v_j|_I$$

is one-to-one for every interval I of length $> 2(\ell_1 + \ell_2 + \cdots)$.

Proof of the theorem

We may write the solutions in the form

$$u_j(t,r)=\sum_{k=1}^{\infty}\left(b_{j,k}e^{i\omega_{j,k}t}+b_{j,-k}e^{-i\omega_{j,k}t}\right)\sin\left(\frac{k\pi r}{\ell_j}\right),\quad j=1,2,\dots$$

with suitable complex coefficients $b_{j,\pm k}$. Thanks to the hypothesis

$$(j_1,k_1)\ne(j_2,k_2)\Longrightarrow\omega_{j_1,k_1}\ne\omega_{j_2,k_2}$$

we may combine all exponents $\omega_{j,k'}$ into a unique increasing sequence (ω_k). Setting $U_k:=v_j$ if $\omega_k=\pm\omega_{j,k'}$, we may apply the second abstract Ingham type theorem if we show that (ω_k) has a finite upper density

$$D^+\le\frac{1}{\pi}\sum_{j=1}^{\infty}\ell_j.$$

Estimate of D^+

Each set

$$\left\{\pm\frac{k\pi}{\ell_j}\ :\ k=1,2,\ldots\right\}$$

has density ℓ_j/π. Since $\left(\omega_{j,k}-\frac{k\pi}{\ell_j}\right)\to 0$ as $k\to\infty$, each set

$$\{\omega_{j,k}\ :\ k=1,2,\ldots\}$$

also has density ℓ_j/π, and hence their union has an upper density

$$D^+\le\frac{1}{\pi}\sum_{j=1}^{\infty}\ell_j.$$

Observability of the beam system

We recall the system of beams that we study:

$$\begin{cases} u_{j,tt} + u_{j,rrrr} = 0 & \text{in} \quad \mathbb{R} \times (0, \ell_j), \\ u_j(t,0) = u_j(t,\ell_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_{j,rr}(t,0) = u_{j,rr}(t,\ell_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_j(0,r) = u_{j0}(r) & \text{for} \quad r \in (0,\ell_j), \\ u_{j,t}(0,r) = u_{j1}(r) & \text{for} \quad r \in (0,\ell_j) \end{cases}$$

for $j = 1, 2, \ldots$. We want to show that the map

$$(u_{10}, u_{11}, u_{20}, u_{21}, \ldots) \mapsto \sum_{j=1}^{\infty} u_{j,r}(\cdot, 0) v_j|_I$$

is one-to-one if

$$|I| > 2\pi \limsup_{r\to\infty} \frac{1}{r} \sum_{j=1}^{\infty} \left[\frac{\sqrt{r}\ell_j}{\pi} \right].$$

Proof of the theorem

Now we may write the solutions in the form

$$u_j(t, r, \varphi_j, \theta_j) = \sum_{k=1}^{\infty} \left(b_{j,k} e^{i\omega_{j,k} t} + b_{j,-k} e^{-i\omega_{j,k} t} \right) \sin \left(\frac{k\pi r}{\ell_j} \right), \quad j = 1, 2, \ldots$$

with $\omega_{j,k} = \left(\frac{k\pi}{\ell_j} \right)^2$ instead of $\omega_{j,k} = \frac{k\pi}{\ell_j}$. Since

$$(\ell_j/\ell_m)^2 \quad \text{is irrational for all} \quad j \neq m$$

by our assumption, we have

$$(j_1, k_1) \neq (j_2, k_2) \Longrightarrow \omega_{j_1,k_1} \neq \omega_{j_2,k_2},$$

and we may repeat the proof given for the string system if we show that the combined sequence (ω_k) has a finite upper density

$$D^+ \leq \limsup_{r \to \infty} \frac{1}{r} \sum_{j=1}^{\infty} \left[\frac{\sqrt{r}\ell_j}{\pi} \right].$$

Estimate of D^+

For any given $r > 0$ and $j = 1, 2, \dots$, the interval $[-r, r]$ contains at least as many elements of the set

$$\left\{ \pm \left(\frac{k\pi}{\ell_j} \right)^2 \ : \ k = 1, 2, \dots \right\}$$

as any other interval of length $2r$. Therefore

$$n^+(2r) = 2 \sum_{j=1}^{\infty} \left[\frac{\sqrt{r}\ell_j}{\pi} \right],$$

and hence

$$D^+ = \lim_{r \to \infty} \frac{n^+(2r)}{2r} \le \lim_{r \to \infty} \frac{1}{r} \sum_{j=1}^{\infty} \left[\frac{\sqrt{r}\ell_j}{\pi} \right]$$

as required.

Sobolev spaces of fractional order

In order to get explicit estimates, for any fixed $s \in \mathbb{R}$ we denote by $D^s(0, \ell_j)$ the Hilbert spaces obtained by completion of the linear hull of

$$e_{j,k}(x) := \sqrt{2/\ell_j} \sin\left(\frac{k\pi x}{\ell_j}\right), \quad j, k = 1, 2, \ldots$$

with respect to the Euclidean norm

$$\left\|\sum_{k=1}^{\infty} c_k e_{j,k}\right\|_{D^s(0,\ell_j)} := \left(\sum_{k=1}^{\infty} \left(\frac{k\pi}{\ell_j}\right)^{2s} |c_k|^2\right)^{1/2}.$$

For example,

$$D^0(0, \ell_j) = L^2(0, \ell_j), \quad D^1(0, \ell_j) = H_0^1(0, \ell_j) \quad D^{-1}(0, \ell_j) = H^{-1}(0, \ell_j)$$

with equivalent norms.

Effective observability of string systems

Theorem

Consider the infinite string system with $a_j = 0$ for all j. If there exists a constant $A > 0$ such that

$$\operatorname{dist}\left(k\frac{\ell_m}{\ell_j}, \mathbb{Z}\right) \ge \frac{A\ell_j\ell_m}{|k|} \quad \text{for all} \quad m \ne j \quad \text{and} \quad k \ge 1, \qquad (1)$$

then there exists a real number $s \le 0$ such that

$$\sum_{j=1}^{\infty} \frac{1}{\ell_j}\left(\|u_{j0}\|^2_{D^s(0,\ell_j)} + \|u_{j1}\|^2_{D^{s-1}(0,\ell_j)}\right) \le c_s(I)\int_I \left|\sum_{j=1}^{\infty} u_{j,r}(t,0,\varphi_j,\theta_j)\right|^2 dt$$

for every bounded interval I of length $|I| > 2\sum_{j=1}^{\infty}\ell_j$, with some constant $c_s(I) > 0$.

Comments on the hypothesis (1)

The condition (1) implies that ℓ_m/ℓ_j is irrational for all $m \neq j$.

Example

The sequence $(\ell_j) := \left(\frac{1}{2^j+\sqrt{2}}\right)$ satisfies (1).

To give a more general example, we recall that a *Perron number* is an algebraic integer q which is real and exceeds 1, but such that its algebraic conjugates are all less than q in absolute value. For example, the Golden Ratio $q \approx 1.618$ and more generally the Pisot and Salem numbers are Perron numbers.

Proposition

If q is a quadratic Perron number, then the sequence $(\ell_j) := (q^{-j})$ satisfies (1)*.*

Effective observability of the string system

The combined sequence (ω_k) is *not* uniformly separated, but if $\left|\omega_{j,m} - \omega_{k,n}\right|$ is small, then $\left|\omega_{j,m}\right|$ and $\left|\omega_{k,n}\right|$ have to be large. The following, more precise lemma yields the theorem:

Lemma

Assume that

$$\operatorname{dist}\left(k\frac{\ell_m}{\ell_j}, \mathbb{Z}\right) \ge \frac{A\ell_j\ell_m}{|k|} \quad \textit{for all} \quad m \ne j \quad \textit{and} \quad k \ge 1.$$

Then

$$0 < \left|\omega_{j,m} - \omega_{k,n}\right| < \min_j \frac{\pi}{\ell_j} \Longrightarrow \left|\omega_{j,m} - \omega_{k,n}\right| \ge \frac{A\pi^2}{\left|\omega_{j,m}\right|}.$$

Proof of the lemma

We have

$$\begin{aligned}|\omega_{j,m}-\omega_{k,n}| &= \pi\left|\frac{m}{\ell_j}-\frac{n}{\ell_k}\right| = \frac{\pi}{\ell_k}\left|m\frac{\ell_k}{\ell_j}-n\right| \\ &\ge \frac{\pi}{\ell_k}\,\text{dist}\left(m\frac{\ell_k}{\ell_j},\mathbb{Z}\right) \\ &\ge \frac{A\pi\ell_j}{|m|} = \frac{A\pi^2}{|\omega_{j,m}|}.\end{aligned}$$

Effective observability of beam systems

Theorem

If there exists an $A > 0$ such that

$$(\ell_j/\ell_m)^2 \notin \mathbb{Q} \quad \textit{and} \quad \operatorname{dist}\left(k\frac{\ell_m}{\ell_j}, \mathbb{Z}\right) \geq \frac{A\ell_j\ell_m}{|k|}$$

for all $m \neq j$ and $k \geq 1$, then

$$\sum_{j=1}^{\infty} \frac{1}{\ell_j}\left(\|u_{j0}\|^2_{H^1_0(0,\ell_j)} + \|u_{j1}\|^2_{H^{-1}(0,\ell_j)}\right) \leq c_s(I)\int_I \left\|\sum_{j=1}^{\infty} u_{j,r}(t,0,\varphi_j,\theta_j)v_j\right\|^2 dt$$

for every bounded interval I of length

$$|I| > 2\pi \limsup_{r\to\infty} \frac{1}{r}\sum_{j=1}^{\infty}\left[\frac{\sqrt{r}\ell_j}{\pi}\right].$$

Idea of the proof

We have to show that the combined sequence $(\omega_k)_{k=-\infty}^{\infty}$ is uniformly separated. Since $\ell_j \to 0$, there is a maximal length ℓ_k, and therefore

$$\inf\left\{\omega_{j,m} \ : \ j, m \ge 1\right\} = \inf_{j\ge 1} \inf_{m\ge 1} \left(\frac{m\pi}{\ell_j}\right)^2 = \left(\frac{\pi}{\ell_k}\right)^2 > 0.$$

By symmetry it remains to show that

$$\left(\frac{m\pi}{\ell_j}\right)^2 - \left(\frac{n\pi}{\ell_k}\right)^2 \ge \min\left\{A\pi^2, \min_j \frac{3\pi^2}{\ell_j^2}\right\} \quad \text{whenever} \quad \frac{m\pi}{\ell_j} > \frac{n\pi}{\ell_k} > 0.$$

If $j = k$, then this follows from the inequality

$$m^2 - n^2 = (m+n)(m-n) \ge m+n \ge 3.$$

If $j \ne k$, then using a preceding lemma we have

$$\left(\frac{m}{\ell_j}\right)^2 - \left(\frac{n}{\ell_k}\right)^2 = \left(\frac{m}{\ell_j} + \frac{n}{\ell_k}\right) \cdot \left(\frac{m}{\ell_j} - \frac{n}{\ell_k}\right) \ge \frac{m}{\ell_j} \cdot \frac{A\ell_j}{m} = A.$$

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