On self-intersections of Laurent polynomials

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(based on joint work with L.V. Kovalev)

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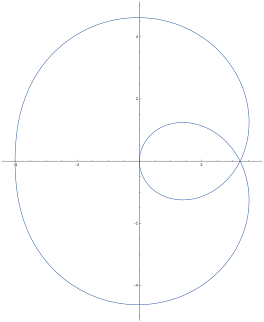
Let P be an algebraic polynomial of degree n.

Definition 1

We say that w is a vertex of the curve $P(e^{i\varphi})$, $0 \leq \varphi \leq 2\pi$, if there exist distinct points z_1 and z_2 on the unit circle $|z| = 1$ such that $p(z_1) = p(z_2) = w$.

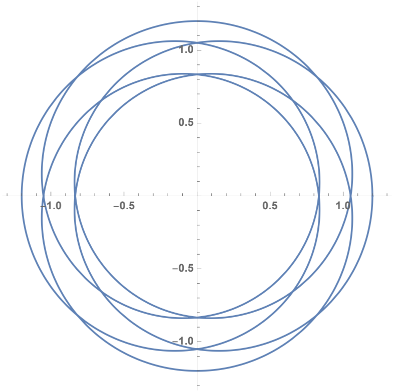


$$P(z) = z^3 - 2z^2 - 3z$$

In 1961, C.J. Titus formulated the following[1]

Conjecture

With certain exceptions the curve $P(e^{i\varphi})$ has at most $(n-1)^2$ vertices.



$$P(z) = z^5 + 0.2z$$

[1] C.J. Titus, The combinatorial topology of analytic functions on the boundary of a disk, Acta Math. 106 (1961),45–64.

In 1973, J. R. Quine confirmed the conjecture by proving the following[2]

Theorem

If P is a polynomial of degree n and not of the form $G(z^m)$, where G is a polynomial and m is an integer , $m > 1$, then the curve $P(e^{i\varphi})$, $0 \leq \varphi \leq 2\pi$, has at most $(n-1)^2$ vertices. Furthermore this bound is sharp, i.e. for any integer $n > 1$ we can find a polynomial P of degree n such that $P(e^{i\varphi})$ has exactly $(n-1)^2$ vertices.

[2] J.R. Quine. On the self-intersections of the image of the unit circle under a polynomial mapping, Proc. Amer. Math. Soc.39(1973), 135–140.

We consider a Laurent polynomial

$$p(z) = \sum_{k=m}^{n} a_k z^k, \quad z \in \mathbb{C} \setminus \{0\},$$

where $m, n \in \mathbb{Z}$, $a_m \neq 0$, and $a_n \neq 0$.

As above, we are interested in the self-intersections of the closed parametric curve $p(\mathbb{T}) = \{p(e^{i\theta})\colon 0 \leq \theta \leq 2\pi\}$.[3]

Remark. Replacing φ by $-\varphi$, we make sure that $n \geq |m|$. Also, since the constant term does not affect on self-intercections, we may assume $m \neq 0$.

[3] S.Kalmykov, L.V. Kovalev. Self-intersections of Laurent polynomials and the density of Jordan curves. Proc. Amer. Math. Soc. (accepted).

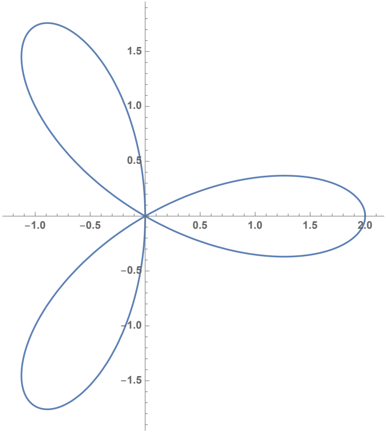
Definition 1'

Self-intersection of p on $\mathbb{T}$ is a two-point subset $\{z_1, z_2\} \subset \mathbb{T}$ where $z_1 \neq z_2$ and $p(z_1) = p(z_2)$.

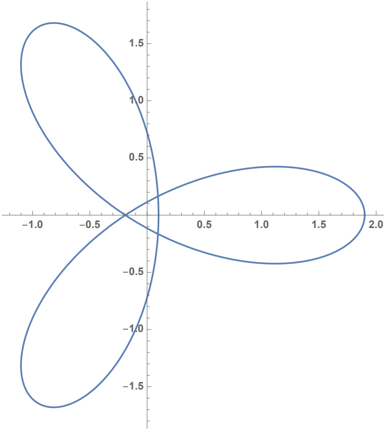
Example

The image of $\mathbb{T}$ under $p(z) = z^2 + z^{-1}$ passes through 0 three times, which counts as three self-intersections, namely $\{e^{\pi i/3}, -1\}$, $\{e^{-\pi i/3}, -1\}$, and $\{e^{\pi i/3}, e^{-\pi i/3}\}$.

Remark. To motivate this way of counting, observe that the image of $\mathbb{T}$ under a perturbed function $z^2 + cz^{-1}$ with c close to 1 has three distinct self-intersections near 0.



$$L(z) = z^2 + 1/z$$



$$L(z) = z^2 + 0.9/z$$

Theorem (K.-Kovalev)

If $-n \leq m < n$ and $m \neq 0$, the number of self-intersections of the Laurent polynomial p on $\mathbb{T}$ is at most

$$\begin{cases} (n-1)\left(n - \frac{m+1}{2}\right), & 1 \leq m < n \\ (n-1)(n-m), & -n < m \leq -1 \\ (n-1)(2n-1), & m = -n \end{cases}$$

with the following exceptions: (a) p can be written as $q(z^j)$ for some Laurent polynomial q and some integer $j \neq -1, 1$; (b) $n = -m$ and $|a_n| = |a_m|$.

Remark. If $p = q(z^j)$ with $j \neq -1, 1$, the polynomial p traces a closed curve more than once, thus creating uncountably many self-intersections. If $n = -m$ and $|a_n| = |a_m|$, the number of self-intersections may also be infinite: consider $p(z) = q(z + 1/z)$ where q is an algebraic polynomial of degree n. This polynomial has self-intersections $p(z) = p(1/z)$, for all $z \in \mathbb{T} \setminus \{-1, 1\}$.

Let U_n, $n \in \mathbb{N}$, be the Chebyshev polynomial of the second kind of degree n.

Recall that

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

By convention, $U_{-1} \equiv 0$ and $U_{-n-1} = -U_{n-1}$ for $n \in \mathbb{N}$.

Lemma 1

Consider a Laurent polynomial with $-n < m < n$, $m \neq 0$, $a_n \neq 0$, and $a_m \neq 0$. Let

$$g(t,z) = \sum_{k=m}^{n} a_k U_{k-1}(t) z^{k-m},$$

$$g^*(t,z) = z^{n-m}\overline{g(\bar{t}, 1/\bar{z})} = \sum_{k=m}^{n} \overline{a}_k U_{k-1}(t) z^{n-k}.$$

Then, with $t = \cos\varphi$, we have

$$g(t,z) = z^{-m}\frac{p(e^{i\varphi}z) - p(e^{-i\varphi}z)}{e^{i\varphi} - e^{-i\varphi}}.$$

Also, g is a polynomial in t, z of total degree $2n - m - 1$, and g^* is a polynomial in t, z of total degree $n - m + |m| - 1$. Finally, if g and g^* are considered as elements of $\mathbb{C}[t][z]$, their resultant is a polynomial of degree $2(n-1)(n-m)$ in t.

$$\operatorname{res}(g, g^*) = \det \begin{pmatrix} a_n U_{n-1} & \cdots & a_m U_{m-1} & \\ \ddots & & \ddots & \\ & a_n U_{n-1} & \cdots & a_m U_{m-1} \\ \overline{a_m} U_{m-1} & \cdots & \overline{a_n} U_{n-1} & \\ \ddots & & \ddots & \\ & \overline{a_m} U_{m-1} & \cdots & \overline{a_n} U_{n-1} \end{pmatrix}$$

Lemma 2

Let p, g, g^* be as in the previous lemma. Given a self-intersection of $p|_{\mathbb{T}}$, write it in the form $\{e^{i\varphi}z, e^{-i\varphi}z\}$ where $z \in \mathbb{T}$ and $e^{i\varphi} \in \mathbb{T} \setminus \{-1, 1\}$. Let $t = \cos\varphi$. Then

$$g(t, z) = g^*(t, z) = g(-t, -z) = g^*(-t, -z) = 0,$$

i.e., the algebraic curves $g = 0$ and $g^* = 0$ intersect at the points (t, z) and $(-t, -z)$. Different self-intersections correspond to different pairs $\{(t, z), (-t, -z)\}$.

We begin with the case $n = -m$. For $z \in \mathbb{T}$ the Laurent polynomial p agrees with the harmonic polynomial

$$p_h(z) = \sum_{k=1}^{n} (a_k z^k + a_{-k} \bar{z}^k).$$

Let $\psi(w) = w + c\overline{w}$, where $c = -a_{-n}/\overline{a_n}$. Then, $\psi^{-1}(\zeta) = (\zeta - c\overline{\zeta})/(1 - |c|^2)$. We have

$$\psi \circ p_h(z) = \sum_{k=1}^{n} ((a_k + c\overline{a_{-k}})z^k + (a_{-k} + c\overline{a_k})\bar{z}^k),$$

where the coefficient of $\bar{z}^n$ vanishes by the choice of c. Returning to the Laurent polynomial form, we have for $z \in \mathbb{T}$,

$$\psi \circ p(z) = \sum_{k=1}^{n} (a_k + c\overline{a_{-k}})z^k + \sum_{k=1}^{n-1} (a_{-k} + c\overline{a_k})z^{-k}.$$

If $\psi \circ p_{|\mathbb{T}}$ depends only on z^j for some $j \in \mathbb{Z} \setminus \{-1, 1\}$, then by applying the inverse transformation ψ^{-1} we conclude that the original polynomial p had the same property, i.e., exceptional case (a) holds. Apart from this exceptional case, we can apply Theorem to $\psi \circ p$, with $m > -n$. The bound is largest when $m = 1 - n$, when it is equal to $(n-1)(2n-1)$. This completes the case $n = -m$.

From now on, $-n < m < n$. g and g^* are relatively prime in $\mathbb{C}[t, z]$. By Bezout's theorem they have at most $\deg g \, \deg g^*$ common zeros. By Lemma 2, the number of self-intersections of $p_{|\mathbb{T}}$ is at most $\frac{1}{2} \deg g \, \deg g^*$. This proves the case $m \geq 1$.

The case $m \le -1$ requires additional consideration of the intersection between $g = 0$ and $g^* = 0$ at infinity.

To do this we can write the polynomials g and g^* in terms of homogeneous coordinates (t, z, w) as follows:

$$G(t, z, w) = w^{2n-m-1} g(t/w, z/w) = \sum_{k=m}^{n} a_k U_{k-1}(t/w) z^{k-m} w^{2n-k-1}$$

and

$$G^*(t, z, w) = w^{n-2m-1} g^*(t/w, z/w) = \sum_{k=m}^{n} a_k U_{k-1}(t/w) z^{n-k} w^{k-2m-1}.$$

So, we have

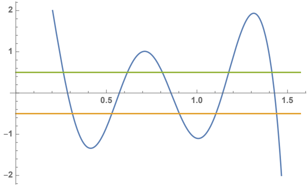
$$(2n-m-1)(n-2m-1) - 2m(m-n) - (n-1)(-m-1) = 2(n-1)(n-m).$$

Proposition (K.-Kovalev)

Suppose $n, m \in \mathbb{Z}$, $n > -m \geq 1$, and $\gcd(n, m) = 1$. Then for sufficiently small $\epsilon > 0$ the Laurent polynomial $p(z) = z^n + \epsilon z^m$ has $(n-1)(n-m)$ self-intersections on $\mathbb{T}$.

$$g(t, z) = U_{n-1}(t) z^n + \epsilon U_{m-1}(t) z^m = \left(\frac{\sin n\theta}{\sin m\theta} z^{n-m} + \epsilon \right) \frac{\sin m\theta}{\sin \theta} z^m$$

where $t = \cos\theta$.

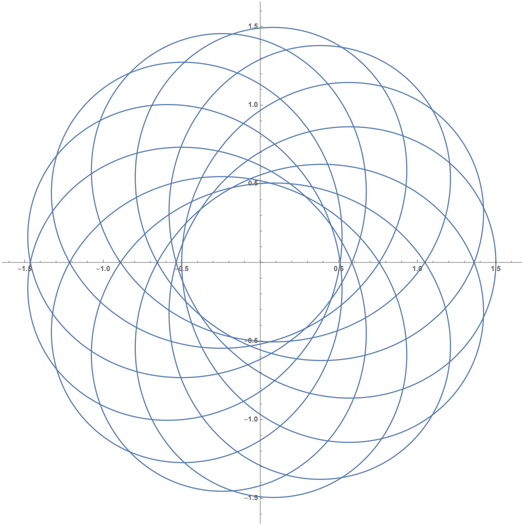


$n = 11, \ m = -2$

Note that U_{n-1} and U_{m-1} have no common zeros because $\gcd(n, m) = 1$. Therefore, any solution of $g(t, z) = 0$ with $|z| = 1$ and $0 < t < 1$ arises from

(1) $$\frac{\sin n\theta}{\sin m\theta} = \pm\epsilon, \quad 0 < \theta < \frac{\pi}{2}.$$

The zeros of the left-hand side of (1) on $[0, \pi/2]$ are $\pi k/n$ for $1 \le k \le \lfloor n/2 \rfloor$. It follows that for small enough ϵ, (1) holds at $n - 1$ points of $(0, \pi/2)$. Indeed, there are two such points near $\pi k/n$ with $1 \le k < \lfloor n/2 \rfloor$, and one such point next to $\pi/2$ (only if n is even). This adds up to $2(n/2 - 1) + 1 = n - 1$ when n is even, and $2(n - 1)/2 = n - 1$ when n is odd.

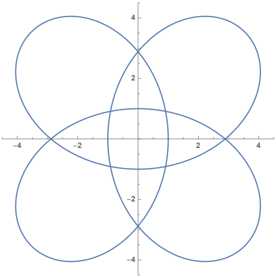


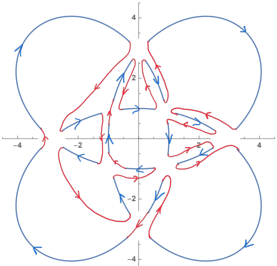
$$L(z) = z^{11} + 0.5/z^2, \ \sharp = 130$$

Let $\mathcal{E}(\mathbb{T};\mathbb{C})$ be the set of all circle embeddings, i.e., continuous injective maps of $\mathbb{T}$ into $\mathbb{C}$. It is well known that continuous maps are dense in $L^p(\mathbb{T};\mathbb{C})$ for $1 \le p < \infty$.

Theorem (K.-Kovalev)

For $p \in [1,\infty)$, every function $f \in L^p(\mathbb{T};\mathbb{C})$ can be approximated in the L^p norm by orientation-preserving C^∞-smooth embeddings of $\mathbb{T}$ into $\mathbb{C}$.





Note that the real-variable analog of this result is false: continuous injective maps $f : [0,1] \to \mathbb{R}$ are not dense in $L^p([0,1])$ for any p, as their closure is the set of monotone functions.

Also, $\mathcal{E}(\mathbb{T};\mathbb{C})$ is not dense in the space of continuous maps $C^0(\mathbb{T},\mathbb{C})$ with the uniform norm, e.g., a continuous map of $\mathbb{T}$ onto a "figure eight" curve cannot be uniformly approximated by injective maps.

The Fourier coefficients of an integrable function $f : \mathbb{T} \to \mathbb{C}$ are given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta.$$

The previous theorem and Parseval's theorem imply the following result.

Corollary

For any sequence $c \in \ell^2(\mathbb{Z})$ and any $\epsilon > 0$ there exists an oriantation-preserving circle embedding $f : \mathbb{T} \to \mathbb{C}$ such that $\|c - \hat{f}\|_2 < \epsilon$.

Thank you for your attention!