Moving and oblique observations of beams and plates

Philippe Jaming

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Budapest, August 2019

Joint work with V. Komornik (Strasbourg)

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General theme: inverse problems of the following form:

$u \in X$ unknown

$Tu \in Y$ a transform (physical system, e.g. sol of a PDE) *inversitble*

Tu only partially known

Is u still uniquely determined?

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Theorem (Sjölin/Kellay-J.)

u sol. of Schrödinger equation

$$\partial_t u(t,x) + i\partial_x^2 u(t,x) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}$$

+ $u_0 := u(0,x)$, $\hat{u}_0 \in L^1(\mathbb{R})$
$u(t,at) = u(t,bt) = 0 \Rightarrow u = 0$.

But recovery of u_0 from $u(t,at) = u(t,bt) = 0$ (most likely) unstable

$$\int_{\mathbb{R}} |u_0(x)|^2 dx \lesssim \int_{\mathbb{R}} |u(t,at)|^2 + |u(t,bt)|^2 dt$$

does not hold...

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$$\begin{cases} u_t + iu_{xx} = 0 & \text{in } \mathbb{R}\times(0,2\pi), \\ u(t,0) = u(t,2\pi) & \text{for } t\in\mathbb{R}, \\ u_x(t,0) = u_x(t,2\pi) & \text{for } t\in\mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x\in(0,2\pi). \end{cases} \tag{1}$$

$u_0 \in L^2(0,2\pi) \Rightarrow u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}$.

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extend as 2π-periodic in x.

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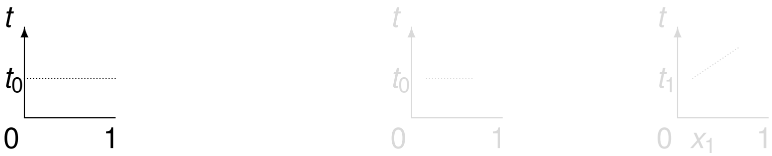

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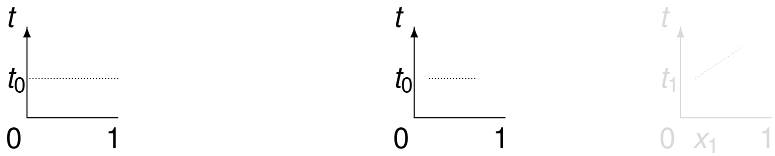

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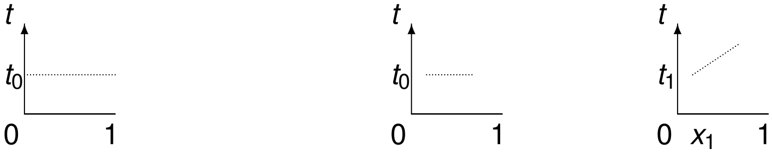

1D periodic Schrödinger - Main result

Theorem (J.-Komornik)

$(t_1, x_1) \in \mathbb{R}^2$, $a \in \mathbb{R}$ and $T > 0$, $u_0 \in L^2(I)$.

- *Direct inequality*

$$\int_0^T |u(t_1 + t, x_1 - at)|^2\,dt \ll \sum_{k\in\mathbb{Z}} |c_k|^2.$$

- *$a \notin \mathbb{Z}$: inverse inequality*

$$\sum_{k\in\mathbb{Z}} |c_k|^2 \ll \int_0^T |u(t_1 + t, x_1 - at)|^2\,dt \qquad (2)$$

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$\Lambda \subset \mathbb{R}^d$, uniformly separated if

$$\gamma(\Lambda) := \inf \{|\lambda_1 - \lambda_2| \ : \ \lambda_1, \lambda_2 \in \Lambda \text{ and } \lambda_1 \neq \lambda_2\} > 0$$

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$$\begin{aligned} u(t_1+t, x_1-at) &= \sum_{k\in\mathbb{Z}} c_k e^{i(k^2(t_1+t)+k(x_1-at))} \\ &= \sum_{k\in\mathbb{Z}} d_k e^{i(k^2-ak)t}. \end{aligned}$$

$$\Lambda = \{k^2 - ak : k \in \mathbb{Z}\} = \{(k^2 - a/2)^2 - a^2/4 : k \in \mathbb{Z}\} = \underbrace{\Lambda_+}_{k\geq a/2} \cup \underbrace{\Lambda_-}_{k<a/2}$$

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- Case 1: $a \notin \mathbb{Z}$ Λ uniformly discrete

$$|(k^2 - ak) - (m^2 - am)| = |k - m||k + m - a| \geq dist(a, \mathbb{Z})$$

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- Case 1: $a \notin \mathbb{Z}$ Λ uniformly discrete

$$|(k^2 - ak) - (m^2 - am)| = |k - m||k + m - a| \geq dist(a, \mathbb{Z})$$

- Case 2: $a \in \mathbb{Z}$

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Fixing lack of uniqueness

Theorem (J.-K.)

$T > 0$, $(t_1, x_1), (t_2, x_2) \in \mathbb{R}^2$, $a_1 \neq a_2 \in \mathbb{Z}$

(i)

$$u(t_1 + t, x_1 - a_1 t) = u(t_2 + t, x_2 - a_2 t) = 0 \quad t \in (0, T),$$

then $u_0 = 0$.

$$\sum_{k\in\mathbb{Z}} |c_k|^2 \ll \int_0^T |u(t_1 + t, x_1 - a_1 t)|^2 + |u(t_2 + t, x_2 - a_2 t)|^2 \, dt$$

fails.

(ii) $(t_1, x_1) \in \mathbb{R}^2$, $a_1, \ldots, a_m \in \mathbb{Z}$. *This also fails*

$$\sum_{k\in\mathbb{Z}} |c_k|^2 \ll \sum_{i=1}^m \int_0^T |u(t_1 + t, x_1 - a_i t)|^2 \, dt$$

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Proof

$u(t_1 + t, x_1 - at) = 0 \Rightarrow d_k + d_{a-k} = 0$ with $d_k := c_k e^{i(k^2 t_1 + k x_1)}$

$|c_k| = |d_k| = |d_{a-k}| = |c_{a-k}|$.

$u(t_2 + t, x_2 - bt) = 0 \Rightarrow |c_k| = |c_{b-k}|$.

$\Rightarrow |c_k| = |c_{a-(a-b+k)}| = |c_{a-b+k}|$.

Contradicts $|c_k| \in \ell^2$ unless $c_k = 0\ \forall k$.

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Beam equation

$$\begin{cases} u_{tt} + u_{xxxx} = 0 & \text{in } \mathbb{R} \times (0, 2\pi), \\ u(t,0) = u(t,2\pi) & \text{for } t \in \mathbb{R}, \\ u_x(t,0) = u_x(t,2\pi) & \text{for } t \in \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x \in (0,2\pi), \quad u_0 \in H^2 \\ u_t(0,x) = u_1(x) & \text{for } x \in (0,2\pi) \quad u_1 \in L^2. \end{cases}$$

$$u(t,x) = c_0^+ + c_0^- t + \sum_{k \in \mathbb{Z}^*} \left(c_k^+ e^{i(k^2 t + kx)} + c_k^- e^{i(-k^2 t + kx)} \right)$$

$$\sum_{k \in \mathbb{Z}} (1 + k^4)\left(\left|c_k^+\right|^2 + \left|c_k^-\right|^2\right) \asymp \|u_0\|_{H_p^2}^2 + \|u_1\|_{L^2}^2.$$

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Beam equation - horizontal segments

Theorem (J.-K.)

$$I := \int_0^X |u(t_1,x)|^2 + |u(t_2,x)|^2\,dx \ll \sum_{k\in\mathbb{Z}} (|c_k^+|^2 + |c_k^-|^2)$$

$$\gtrsim |c_0^+|^2 + |c_0^-|^2 + \sum_{k\in\mathbb{Z}^*} \sin^2 k^2(t_1 - t_2)(|c_k^+|^2 + |c_k^-|^2)$$

1. $(t_2 - t_1)/\pi \in \mathbb{Q}$ $\exists u_0, u_1 \neq 0$ *s.t.* $I = 0$
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Beam equation - oblique segments

$$u(t_0+t, x_0-at) = d_0^+ + d_0^- t + \sum_{k\in\mathbb{Z}^*} \left(d_k^+ e^{i(k^2-ak)t} + d_k^- e^{i(-k^2-ak)t} \right),$$

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A family of circles

$a \in \mathbb{R}$, $S_a \subset \mathbb{R}^2$ circle centered in $(a/2, -a/2)$ through $(0,0)$.

$A_a = S_a \cap \mathbb{Z}^2 \setminus \{(0,0)\}$.

(i) $|A_a| \leq \sqrt{2}\pi|a|$.

(ii) If $a \in \mathbb{Z}^*$, then $(a, -a) \in A_a \neq \emptyset$.

(iii) If $a \notin \mathbb{Q}$, $A_a = \emptyset$: $(k, m) \in A_a \Rightarrow a = \frac{k^2+m^2}{k-m} \in \mathbb{Q}$.

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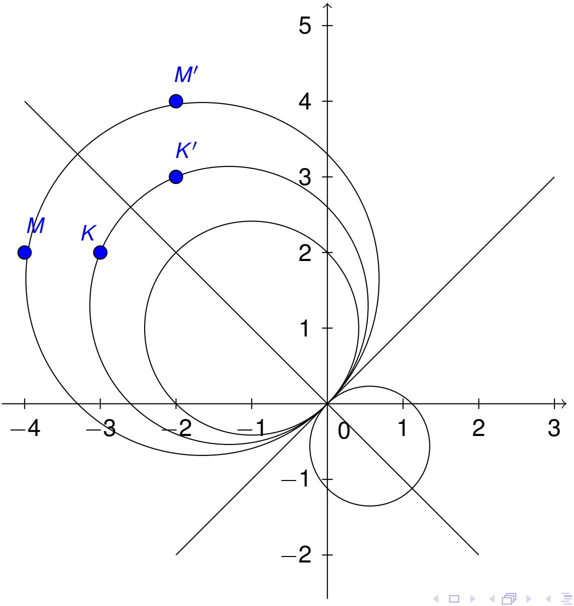
Beam equation: two oblique segments/ a graph



Beam equation: one oblique segment

Theorem

$(t_1, x_1) \in \mathbb{R}^2$, $a \in \mathbb{R}$ *and* $T > 0$

(i) $$\int_0^T |u(t_1 + t, x_1 - at)|^2\,dt \ll \sum_{k\in\mathbb{Z}} \left(|c_k^+|^2 + |c_k^-|^2\right)$$

(ii) *If* $a \neq 0$ *and* $A_a = \varnothing$ *(e.g.* $a \notin \mathbb{Q}$*),*

$$\sum_{k\in\mathbb{Z}} \left(|c_k^+|^2 + |c_k^-|^2\right) \ll \int_0^T |u(t_1 + t, x_1 - at)|^2\,dt$$

(iii) $a \in \mathbb{Z}$ *or* $A_a \neq \varnothing$, $\exists u$ *s.t.* $u(t_1 + t, x_1 - at) = 0$.

Beam equation: one oblique segment

Theorem

$(t_1, x_1) \in \mathbb{R}^2$, $a \in \mathbb{R}$ and $T > 0$

- (i) $$\int_0^T |u(t_1 + t, x_1 - at)|^2 \, dt \ll \sum_{k \in \mathbb{Z}} \left(|c_k^+|^2 + |c_k^-|^2 \right)$$
- (ii) If $a \neq 0$ and $A_a = \varnothing$ (e.g. $a \notin \mathbb{Q}$),
$$\sum_{k \in \mathbb{Z}} \left(|c_k^+|^2 + |c_k^-|^2 \right) \ll \int_0^T |u(t_1 + t, x_1 - at)|^2 \, dt$$
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possible trouble from 3rd term $\to$ get rid of it with a second segment.

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$+\sum_{(k,m)\in A_a}\left|d_k^+ + d_m^-\right|^2$

possible trouble from 3rd term $\to$ get rid of it with a second segment.

Beam equation: one oblique segment

$$u(t_0+t,x_0-at)=d_0^+ + d_0^- t + \sum_{k\in\mathbb{Z}^*}\left(d_k^+ e^{i(k^2-ak)t} + d_k^- e^{i(-k^2-ak)t}\right),$$

Ingham-Kahane $\Rightarrow$

$$\int_0^T |u(t_1+t,x_1-at)|^2\,\mathrm{d}t \asymp$$

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Beam equation: two oblique segments

$$\int_0^T |u(t_1+t, x_1-at)|^2\,\mathrm{d}t + \int_0^T |u(t_1+t, x_1-bt)|^2\,\mathrm{d}t \asymp$$

$\sum_\pm \sum_{k\in(\mathbb{Z}\setminus A_a^\pm)\cup(\mathbb{Z}\setminus A_b^\pm)} \left|d_k^\pm\right|^2$

$+\sum_{(m,n)\in A_a\cup A_b} \left|d_k^+ + d_m^-\right|^2$

We want

$$\sum_{(m,n)\in A_a\cup A_b} \left|d_k^+ + d_m^-\right|^2 \asymp \sum_{(m,n)\in A_a\cup A_b} \left|d_k^+\right|^2 + \left|d_m^-\right|^2$$

These are 2 quadratic forms in the finite number of variables d_k^+, d_m^-, $(k,m)\in A_a\cup A_b$.

Prove the first one is non-degenerate.

Beam equation: two oblique segments

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Beam equation: two oblique segments/ a graph