The regularity of the solutions of the abstract evolution equations of hyperbolic type

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Fruitful comments from Prof. Emeritus Hiroki Tanabe (Osaka Univ.) are acknowledged.

Related works

1) Logarithmic representation of operators

_ alternative generator （~extraction of analytic semigroup ）

_ relativistic formulation

YI, Methods Funct.Anal.Topology 2017
YI, J. Appl. Math. Phys. 2017
YI, AIP Conf. Proc, 2019

2) Algebraic structure of operators

_ A module over the Banach algebra
= “B(X)-module theory”

YI, J.Phys Conf. Ser. 2018
YI, Adv. Math. Phys. 2019 (to appear, invited)

_ Unbounded formulation of the rotation group

YI, J.Phys Conf.Ser, 2019

3) Nonlinearity of operators (quick review)

YI, Methods Funct.Anal.Topology 2019

Introduction

We want to define convergent power series $e^{tA} = \Sigma\,(tA)^n/n!$
for unbounded generator A e.g., Schroedinger operator

◆ **Evolution equation theory** (infinite dimensional)
→ Evolution operator: bounded
→ infinitesimal generator: unbounded → **Hille-Yosida**

◆ **Lie theory or Banach algebra** (finite/infinite dimensional)
→ Evolution operator (Lie group): bounded
→ infinitesimal generator (Lie algebra): (un)bounded
→ **Power series (Campbell-Baker-Hausdorff)**

Abstract Cauchy problem of hyperbolic type

t-dependent non-autonomous problem T. Kato (1970, 1973)

$$(*) \quad \begin{cases} \dfrac{du(t)}{dt} = \boxed{A(t)} u(t) + f(t), \quad 0 \le t \le T, \\ u(0) = u_0 \in X \end{cases}$$

t-dependent

X: Banach space (= **abstract space**)

A(t): infinitesimal generator of $U(t,s) \in X$ **generator of group**

PDEs can be modeled by "ODEs in infinite-dimensional Banach space *X* "

Point of argument

In most of standard theory of evolution equations,
Time evolution is represented by the exponential function of operators

→ Represent the generator by logarithm function of operator

Logarithm as an infinitesimal generator

$v(t) = \mathrm{U}(t)\, u_0$

: unknown function

$$\boxed{\partial_t \mathrm{Log} v = v' v^{-1}}$$

$$\Updownarrow$$

$$dv/dt = [\partial_t \mathrm{Log} v] v$$

"Definition of log" is equivalent to **"Definition of generator"** to some extents

We would start with defining **logs**

Research history of logarithm of operators

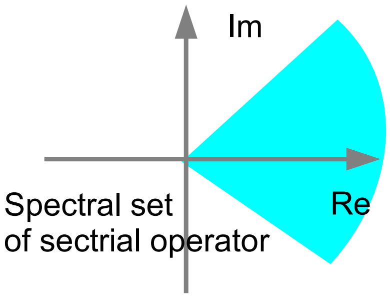
1943 N. Dunford ··· theory of function of operators

1953 A. E. Taylor ··· generalization of Dunford theory

1969 V. Nollau ··· initiation of log of operators

1990's Revival of log in terms of “fractional power”

K. N. Boyadzhiev



2000's N. Okazawa

M. Hasse,

C. Martinez and M. Sanz and so on

These researches are based on the sectorial operator framework.

Here we do not use the sectorial operator framework.

Definition of $U(t,s)$

Evolution operator

Define $U(t,s)$; $t,s \in [-T,T]$ satisfying semigroup property

$$U(t,r)\,U(r,s) = U(t,s)$$

$$U(s,s) = I$$

… associated with Hyperbolic type PDE

Furthermore the inverse is defined as

$$U(s,t)\,U(t,s) = U(s,s) = I$$

As a result $U(t,s)$ is C_0-group for $[-T,T]$

In particular $U(t,s)$ is bounded on X

Continuous group

$$\|U(t,s)\|_{B(X)} \le M e^{\beta t} \qquad t,s \in [-T,T]$$

Logarithm of operator

Dunford-Riesz integral

Principal branch of logarithm

$$\boxed{\mathrm{Log}\, z} = \log|z| + i \arg Z$$

$$|Z| = |z|, \quad \underline{-\pi < \arg Z \leq \pi}$$

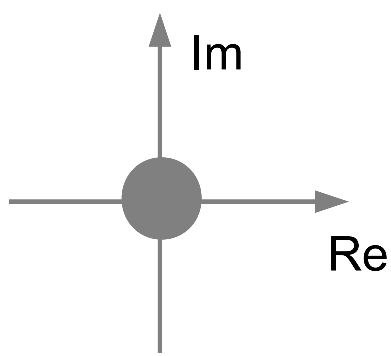
$$\boxed{\partial_t \mathrm{Log} v = v' v^{-1}}$$

Treatment for non-sectorial operator

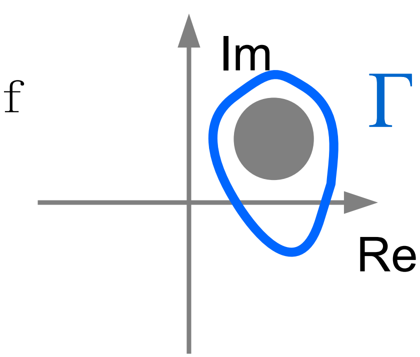
$$\mathrm{Log}(U(t,s) + CI) = \frac{1}{2\pi i} \int_\Gamma \mathrm{Log}\lambda \; (\lambda - U(t,s) - CI)^{-1} d\lambda$$

Draw Γ within the resolvent set of $U(t,s)$ + C

Spectral set of
$U(t,s)$



Spectral set of
$U(t,s)$+C



Derivative of logarithm

Define $A(t): Y \to X$ as

$$\begin{cases} A(t)u := \underset{h\to 0}{\mathrm{wlim}}\, h^{-1}(U(t+h,t)-I)u \\ \boxed{\underset{h\to 0}{\mathrm{wlim}}}\, h^{-1}(U(t+h,s)-U(t,s))\,u = \underset{h\to 0}{\mathrm{wlim}}\, h^{-1}(U(t+h,t)-I)\,U(t,s)\,u \end{cases}$$

Here we consider weak limit
(generalized compared to the standard theory)

Y is a densely defined subspace of X

Remarkable points in defining the logarithm of $A(t)$:

- The origin is the singular point of log
- log is the multi-valued function

Theorem -log representation -

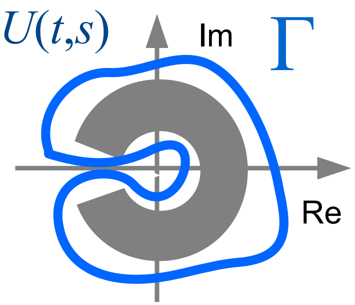
Theorem 3.1. *Let t and s satisfy $-T \le t, s \le T$, and Y be a dense subspace of X. For $U(t,s)$ defined in Sec. 2, let $A(t) \in G(X)$ and $\partial_t U(t,s)$ be determined as the previous sheet. If $A(t)$ and $U(t,s)$ commute, an evolution family $\{A(t)\}_{-T \le t \le T}$ is represented by means of the logarithm function; there exists a certain complex number $C \neq 0$ such that*

$$A(t)\, u = (I + CU(s,t))\, \partial_t \mathrm{Log}\, (U(t,s) + CI)\, u, \tag{10}$$

where u is an element in Y. Note that $U(t,s)$ is assumed to be invertible.

YI, Methods Funct.Anal.Topology 2017

Spectral set of $U(t,s)$



Note: (10) can be valid for some cases with C=0.

Here we consider more general cases:
(10) is valid for $C \neq 0$.
In this settings, all the $U(t,s)$ defined in the preceding sheet is under control.

Outline of the proof

$$\boxed{A(t)u := \operatorname*{wlim}_{h\to 0} h^{-1}(U(t+h,t)-I)u}$$

in order to relate this eq. with logarithm ...

$$\operatorname*{wlim}_{h\to 0}\frac{1}{h}\{\mathrm{Log}\,(U(t+h,s)+CI)-\mathrm{Log}\,(U(t,s)+CI)\}$$

$$=\operatorname*{wlim}_{h\to 0}\frac{1}{h}\frac{1}{2\pi i}\int_\Gamma \mathrm{Log}\lambda$$

$$\{(\lambda-U(t+h,s)-CI)^{-1}-(\lambda-U(t,s)-CI)^{-1}\}d\lambda$$

$$=\operatorname*{wlim}_{h\to 0}\frac{1}{2\pi i}\int_\Gamma \mathrm{Log}\lambda$$

↓ integrated is evaluated in the 1st step

$$\boxed{\{(\lambda-U(t+h,s)-CI)^{-1}\frac{U(t+h,s)-U(t,s)}{h}(\lambda-U(t,s)-CI)^{-1}\}}d\lambda$$

3 steps of the proof

Proof 1st stage → **Proof 2nd stage** → **Proof 3rd stage**

uniformly bounded — Commutation, exchange of integral and limit — Complex calculi

Application to evolution equations

$$U(t,s) = e^{a(t,s)} - CI$$

$$A(t)\; u = (I + CU(s,t))\; \underbrace{\partial_t \mathrm{Log}\; (U(t,s) + CI)}_{:= \partial_t\, a(t,s)}\; u \qquad u \in Y$$

Corollary 3.2. *Let t and s satisfy $0 \le t, s \le T$. For $U(t,s)$ and $A(t)$ satisfying the assumption of Theorem 3.1, the exponential of $a(t,s)$ is represented by a convergent power series:*

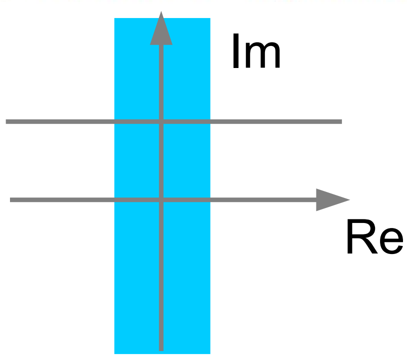
$$e^{a(t,s)} = \sum_{n=0}^{\infty} \frac{a(t,s)^n}{n!}, \tag{15}$$

with a relation $e^{a(t,s)} = \exp(\mathrm{Log}(U(t,s) + CI)) = U(t,s) + CI$. If $a(t,s)$ with different t and s are further assumed to commute,

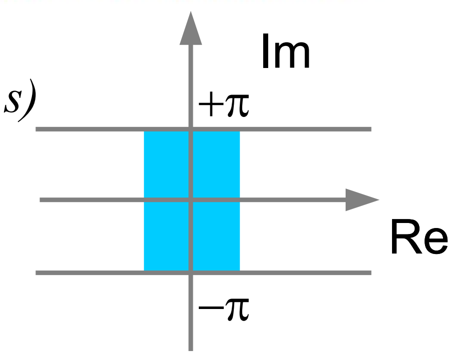
$$\partial_t e^{a(t,s)} u_s = \partial_t a(t,s)\; e^{a(t,s)} u_s \tag{16}$$

is satisfied for $u_s \in Y$, where ∂_t denotes a $t-$differential in a weak sense.

Spectrum of $A(t)$



Spectrum of $a(t,s)$



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Similar to
Maximal regularity
for
Hyperbolic type

Autonomous case $du(t) / dt = A(t)$

Theorem 4.1. *Operator $e^{a(t,s)}$ is holomorphic.*

Theorem 4.2. *For $u_s \in X$ there exists a unique solution $u(t) \in C([-T,T];X)$ of (19) with a convergent power series representation:*

$$(22) \qquad u(t) = U(t,s)u_s = (e^{a(t,s)} - CI)u_s = \left(\sum_{n=0}^{\infty} \frac{a(t,s)^n}{n!} - CI \right) u_s,$$

where C is a certain complex number.

→ Computational advantage is clear

Non-autonomous case $du(t) / dt = A(t) + f(t)$

Theorem 4.3. *Let $f \in L^1(-T,T;X)$ be locally Hölder continuous on $[-T,T]$. For $u_s \in X$ there exists a unique solution $u(t) \in C([-T,T];X)$ for (23) such that*

$$u(t) = [\sum_{n=0}^{\infty} \frac{a(t,s)^n}{n!} - CI]u_s + \int_s^t [\sum_{n=0}^{\infty} \frac{a(t,s)^n}{n!} - CI] f(\tau) d\tau$$

using a certain complex number C.

YI, Methods Funct.Anal.Topology 2017

Connection to Banach algebra

Alternative representation for evolution operator

Group

$$U(t,s) = \boxed{e^{a(t,s)}} - CI$$

Semigroup property

$$U(t,r)U(r,s) = U(t,s),$$
$$U(s,s) = I.$$

[ORIGINAL]
Generated by
unbounded operator
in **infinite** dimensional space

[NEWLY Defined]
Generated by
bounded operator
in **infinite** dimensional space

Generally,
No concrete information

B(X)-module

Semigroup property: YES

$$U(t,s) = \lim_{\lambda \to 0} AI_{\lambda}$$

Semigroup property: NO

$$e^{ta(t,s)} = \Sigma\, (a(t,s))^n/n!$$

YI, Adv. Math. Phys. 2019 (to appear, invited)

Connection to Banach algebra

Alternative representation for evolution operator

Group

$$U(t,s) = \boxed{e^{a(t,s)}} - CI$$

Semigroup property

$$U(t,r)U(r,s) = U(t,s),$$
$$U(s,s) = I.$$

[ORIGINAL]
Generated by
unbounded operator
in **infinite** dimensional space

[NEWLY Defined]
Generated by
bounded operator
in **infinite** dimensional space

More importantly ...

$$U(t,s) = e^{a(t,s)} - CI = \sum_n (a(t,s))^n/n! - CI$$
$$= \sum_n (a(t,s) - \delta_{n,1}\, CI)^n/n!$$

leading to the basis for the Lie group & Lie algebra
(Baker-Hausdorff type formula)

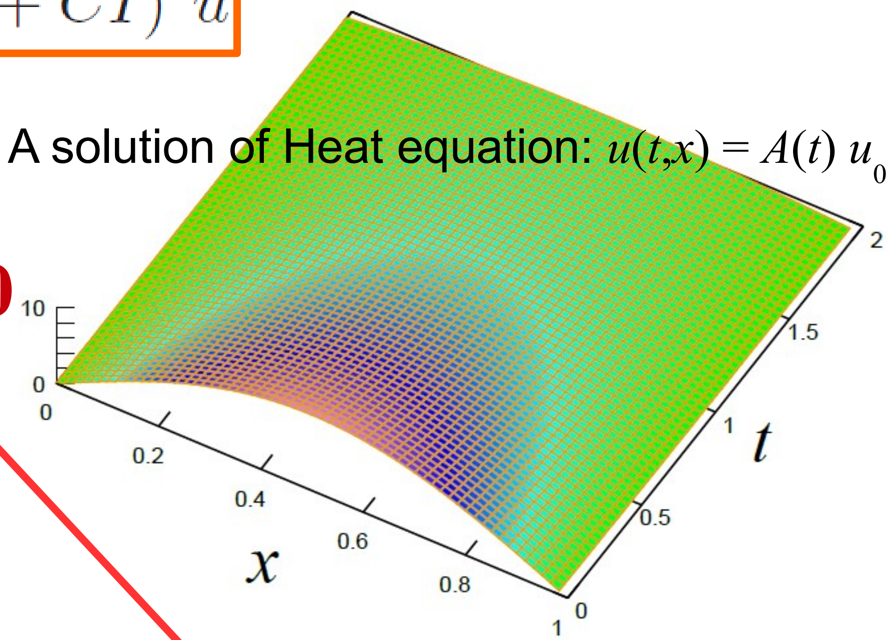
YI, Adv. Math. Phys. 2019 (to appear, invited)

YI, Methods Funct.Anal.Topology 2019

Relation to Cole-Hopf transform

$$A(t)\ u = (I + CU(s,t))\ \partial_t \mathrm{Log}\ (U(t,s) + CI)\ u$$

A solution of Heat equation: $u(t,x) = A(t)\ u_0$



"similar" by assuming C=0

→ space-time replacement

Cole-Hopf transform

$$\partial_t \psi + \psi \partial_x \psi = \mu^{-1/2}\ \partial_x^2 \psi$$

Burgers eq (nonlinear)

$$\psi(t,x) = -2\mu^{-1/2}\ \partial_x \log u(t,x)$$

$$\partial_x^2 u - \mu^{1/2} (-\partial_t^2)^{1/2} u = 0$$

Heat eq. (linear)

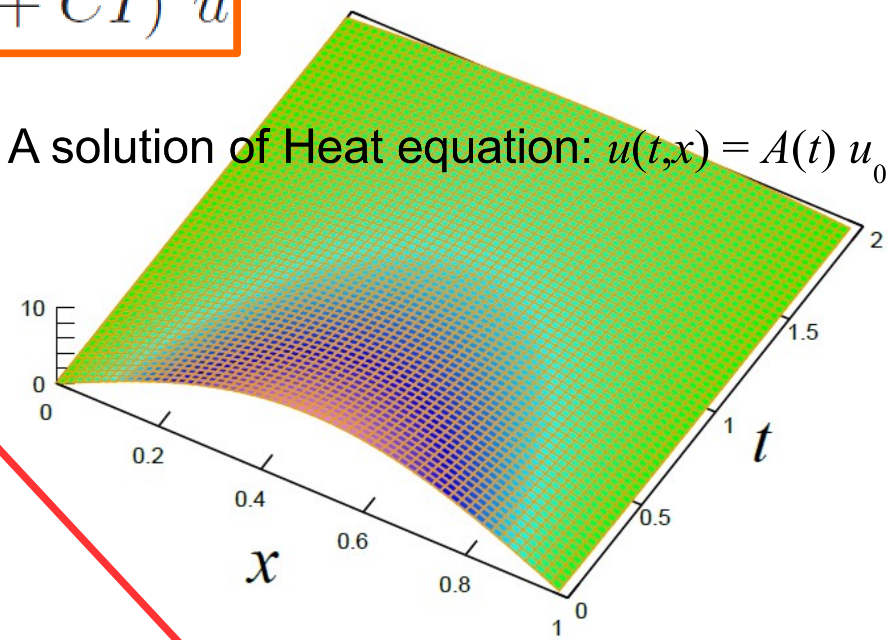
Relation to Cole-Hopf transform

$$\boxed{A(t)\ u = (I + CU(s,t))\,\partial_t \mathrm{Log}\ (U(t,s) + CI)\ u}$$

$$(uv^{-1})' = [\boxed{v'v^{-1}} - \boxed{u'u^{-1}}](uv^{-1})$$

$$\boxed{\partial_t \mathrm{Log} v = v'v^{-1}}$$

A solution of Heat equation: $u(t,x) = A(t)\ u_0$



$$\partial_x^2 u - \mu^{1/2}\partial_t u = 0$$

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & I \\ f\mu^{1/2}\partial_t & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

diagonalization

$$\tilde{A} = \begin{pmatrix} (\mu^{1/2}\partial_t)^{1/2} & 0 \\ 0 & -(\mu^{1/2}\partial_t)^{-1/2} \end{pmatrix}$$

Summary

$$A(t)\ u = (I + CU(s,t))\ \partial_t \underbrace{\mathrm{Log}\ (U(t,s) + CI)}_{:= a(t,s)}\ u$$

$$e^{a(t,s)} = \sum_{n=0}^{\infty} \frac{a(t,s)^n}{n!}$$

- Logarithmic representation of C_0-group is obtained; bounded part of generator is extracted
- [Increased regularity] alternative regularity of hyperbolic type (interface on **the maximal regularity**)
- [Algebraic B(X)-module] foundation to **Lie algebra**
- [Nonlinearity] connection to **the Cole-Hopf transform**.