On the classes of functions of generalized bounded variation

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The notion of bounded variation was based by Jordan[1]. Wiener[2] considered the class of functions BV_p. Love[3] studied functional properties of this class. Young[4] intoduced the notion of Φ-variation. Musielak and Orlicz[5] studied properties of this class. Waterman[6] studied class of functions of bounded Λ-variation. Chanturia[7] defined notion of modulus of variation. Kita and Yoneda[8] introduced new class of functions of bounded variation. T. Akhobadze[9,10] generalized the last class and studied properties of it. This bibliography can be continued.

Definition

Let $f(t)$ be a function defined on a finite closed interval $[a,b]$. Suppose p_n and $\phi(n)$ be sequences such that $p_1 \geq 1$, $p_n \uparrow \infty$, $n \to \infty$ and $\phi(1) \geq 1$, $\phi(n) \uparrow \infty$, $n \to \infty$. We say that $f \in BV(p_n \uparrow \infty, \phi, [a,b])$ if

$$V(f, p_n \uparrow \infty, \phi, [a,b]) = \sup_n \sup_\Delta \left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_n} : \rho(\Delta) \geq \frac{1}{\phi(n)} \right)^{1/p_n} < +\infty,$$

where Δ is $a = t_0 < t_1 < ... < t_m = b$ partition of the interval $[a,b]$ and $\rho(\Delta) = \min_i(t_i - t_{i-1})$.

In the case, $\phi(n)=2^n$, class $BV(p_n\uparrow\infty,\phi,[a,b])$ is considered by Kita and Yoneda [8].

Proposition

$BV(p_n\uparrow\infty,\phi)$ is a linear space and for each numbers α and β we have

$$V(\alpha f+\beta g,p_n\uparrow\infty,\phi)\le|\alpha|V(f,p_n\uparrow\infty,\phi)+|\beta|V(g,p_n\uparrow\infty,\phi).$$

Definition

Denote by $BV^*(p_n\uparrow\infty,\phi,[a,b])$ class of functions from $BV(p_n\uparrow\infty,\phi,[a,b])$ for which $f(a)=0$.

$BV^*(p_n\uparrow\infty,\phi,[a,b])$ is a normed space, with the norme

$$||f||=V(f,p_n\uparrow\infty,\phi,[a,b]).$$

Proposition

$BV^*(p_n \uparrow \infty, \phi, [a,b])$ *is a complete space.*

Proposition

$BV^*(p_n \uparrow \infty, \phi, [a,b])$ *is not separable.*

Proposition

If at each point t of $[a,b]$ interval $\lim_{k\to\infty} f_k(t) = f(t)$, then

$$V(f, p_n \uparrow \infty, \phi) \leq \liminf_{k\to\infty} V(f_k, p_n \uparrow \infty, \phi).$$

Definition

A sequence of f_n functions will be termed convergent in variation to f if $V(f_n - f, p_n \uparrow \infty, \phi) \to 0$ for $n \to \infty$.

Proposition

Convergence in variation implies uniformly convergence, in general.
If $\phi(n)^{\frac{1}{p_n}}$ is bounded then these convergences are equivalent.
If $\phi(n)^{\frac{1}{p_n}}$ is not bounded then there exists uniformly convergent sequence of functions which is not convergent in variation.

Proposition

Let $p_1 \geq 1$, $p_n \uparrow \infty$ *and* $\phi(1) \geq 1$, $\phi(n) \uparrow \infty$. *Then for each point* $x \in (a, b)$ *there exists* $y \in (x, b)$, *and a function* f *defined on* $[a, b]$ *such that*

$$V(f, p_n \uparrow \infty, \phi, [a, y]) < V(f, p_n \uparrow \infty, \phi, [a, x]).$$

Remark

Let f *be defined on* $[a, b]$ *and* $[a_1, b_1] \subset [a, b]$. *then*

$$V(f, p_n \uparrow \infty, \phi, [a_1, b_1]) \leq 3 \cdot V(f, p_n \uparrow \infty, \phi, [a, b]).$$

Remark

If $c \in (a, b)$ *then*

$$V(f, p_n \uparrow \infty, \phi, [a, b]) \leq 4 \cdot V(f, p_n \uparrow \infty, \phi, [a, c]) + 4 \cdot V(f, p_n \uparrow \infty, \phi, [c, b]).$$

Definition

A function f defined on a closed interval $[a,b]$, will be termed $((p_n),\phi)$-absolute continuous if the following condition is satisfied: for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\left(\sum_{i=1}^{m} |f(\beta_i) - f(\alpha_i)|^{p_n}\right)^{1/p_n} < \varepsilon,$$

for all finite sets of non-overlapping intervals $(\alpha_i, \beta_i) \subset [a,b]$, $i = 1, 2, ..., m$, for which $\beta_i - \alpha_i \geq \frac{1}{\phi(n)}$, $i = 1, 2, ..., m$, and

$$\left(\sum_{i=1}^{m} (\beta_i - \alpha_i)^{p_n}\right)^{1/p_n} < \delta.$$

We denote this class by $AC(p_n \uparrow \infty, \phi, [a,b])$.
It is clear that if f is $((p_n),\phi)$-absolute continuous then f is continuous.

Proposition

A necessary and sufficient condition for f *to be in* $AC(p_n \uparrow \infty, \phi, [a,b])$ *is that for a given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$V(f, p_n \uparrow \infty, \phi, [t_1, t_2]) < \varepsilon,$$

for each $[t_1, t_2] \subset [a,b]$ *when* $t_2 - t_1 < \delta$.

Proposition

If f *is absolute continuous (in the ordinary sence), then* $f \in AC(p_n \uparrow \infty, \phi)$.

Proposition

If $\phi(n)^{\frac{1}{p_n}}$ *is bounded then every continuous function on* $[a,b]$ *is* $((p_n), \phi)$*-absolute continuous.*

Proposition

If $\phi(n)^{\frac{1}{p_n}}$ is not bounded then there exists a continuous function f which is not $((p_n), \phi)$-absolute continuous.

Proposition

Let $\{f_k\}_{i=1}^{\infty}$ be a sequence of functions from $AC(p_n \uparrow \infty, \phi, [a, b])$ which is convergent in variation to f, then $f \in AC(p_n \uparrow \infty, \phi, [a, b])$.

Proposition

For every $q \geq 1$ there exists a function which is $((p_n), \phi)$-absolute continuous but it is not in V_q, where V_q is the class of functions of Wiener-Young[4] q-th generalization of total variation.

Definition

We say that a function f defined on $[a,b]$ satisfies condition $(*)$ if it is measurable, periodic with period $b-a$ and if $V(f^h - f, p_n \uparrow \infty, \phi, [a,b]) \to 0$ for $h \to 0+$, where $f^h(t) = f(h+t)$.

Proposition

If a function f is $((p_n), \phi)$-absolute continuous on $[a,b]$, periodic with period $b-a$, then f satisfies condition $(*)$.

Proposition

Let a function f defined on $[a,b]$ satisfies condition $(*)$. Then the sequence f_k of the Steklov functions of f, defined by the formula

$$f_k(t) = k \int_t^{t+\frac{1}{k}} f(\tau)d\tau,$$

is convergent in variation to $f(t)$.

Corollary

Lef f be an integrable, periodic function. Sequence of its Steklov functions is convergent in variation to f if and only if when f is $((p_n), \phi)$ absolute continuous.

Proposition

Let $\int_a^b |k_q(t)|dt = \theta_q$, $q = 1, 2...$, and (θ_q) is bounded; f is $((p_n), \phi)$-absolute continuous, periodic with period $b - a$ and $I_q(t) = \int_a^b K_q(\tau) f(t + \tau) dt$. If for some ξ the sequence of functions $I_q(t)$ converges uniformly to $f^\xi(t)$ then

$$V(I_q - f^\xi, p_n \uparrow \infty, \phi, [a, b]) \to 0, \ q \to \infty,$$

where $f^\xi(t) = f(t + \xi)$.

Corollary

Let f be a periodic function with period 2π and $\sigma_n^\alpha(f)$ be (C,α), $\alpha > 0$, means of Fourier series of f with respect to the trigonometric system. Then $\sigma_n^\alpha(f)$ is convergent in variation to f if and only if $f \in AC(p_n \uparrow \infty, \phi)$.

Corollary

Let $K_q(t) \geq 0$, $\int_a^b K_q(t)dt \to 1$ as $q \to \infty$ and $\int_{a+\delta}^{b-\delta} K_q(t)dt \to 0$ as $q \to \infty$ for each $0 < \delta < \frac{1}{2}(b-a)$ and f is periodic with period $b-a$.
If $f \in AC(p_n \uparrow \infty, \phi, [a,b])$ then $V(I_q - f^a, p_n \uparrow \infty, \phi, [a,b]) \to 0$, where $f^a(t) = f(t+a)$.

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Thank You for Your attention!