Asymptotics for Recurrence Coefficients of X1-Jacobi Polynomials and Christoffel Function

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Preliminaries

$I \subset \mathbb{R}$ - interval, w - weight function on $I \Rightarrow \{p_n\}$ - orthogonal polynomials w.r.t. w on I.

- $\deg p_n = n$, $n = 0, 1, \ldots$, $\mathcal{P}_n := \text{span}\{p_0, \ldots, p_{n-1}\} \subset \mathcal{P}$, polynomials of degree $\leq n-1$.
- p_n has n distinct zeros in I, $z_1, \ldots, z_n$.
- $xp_n = a_{n+1}p_{n+1} + b_n p_n + a_n p_{n-1}$.
- $\pi_n : L^2(w) \longrightarrow \mathcal{P}_n$ with kernel

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y).$$

$$\nu_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}; \quad d\mu_n(x) := \frac{1}{n} K_n(x, x) w(x) dx = \frac{w(x)dx}{n\lambda_n(w, x)}$$

Some results w.r.t. Ordinary Orthogonal Polynomials

- If the recurrence coefficients $a_n \to a > 0$, $b_n \to b$, then

$$\nu_n \to \omega_{a,b}(x)dx = \frac{1}{\pi\sqrt{(b+2a-x)(x-b+2a)}}dx = \mu_e([b-2a, b+2a])$$

(Van Assche)

- supp$\mu \subset C(0,1)$ or supp$\mu \subset [-1,1]$ + Szegő condition, then

$$\mu_n \to \mu_e.$$

(Mátè, Nevai, Totik)

- Szegő condition fulfils on a subinterval + "regularity" (Stahl, Totik)

- On arcs, curves (V.Totik)

- Compact sets (regular to the Dirichlet problem) (B. Simon)

$$(\mu_n - \nu_n) \xrightarrow{w^*} 0.$$

- Real OP ensembles having 3-term recurrence relation (A. Hardy)

Exceptional Orthogonal Polynomials

$$T[y] = py'' + qy' + ry = \lambda y$$

$\lambda_0, \lambda_1, \ldots$ eigenvalues, $p_0, p_1, \ldots$ polynomial eigenfunctions, $\deg p_n = n$, $n = 0, 1, \ldots$

$$\Downarrow$$

p, q, r: polynomials; p_n: Hermite, Laguerre, Jacobi (Bochner)

? $\deg p_n \geq n$, $n = 0, 1, \ldots$?

Definition.

Exceptional polynomials means a co-finite real-valued polynomial sequence $\{p_k\}$, where from the sequence of degrees finite indices $(k_1, \ldots, k_m)$ are missing provided

(1) The polynomials are eigenfunctions of a differential operator of second order with rational coefficients.

(2) There is an interval I and a positive weight function W on I with finite moments and at the endpoints of I $pp_kW \to 0$, where p is the coefficient of the second derivative in the differential operator.

(3) The vector space spanned by the elements of the sequence is dense in the weighted space $L^2(W, I)$.

Construction. (García-Ferrero, Gómez-Ullate, Milson, JMAA, 2019.)

All families of exceptional polynomials can be got from the classical ones ($P_n^{[0]}$) by finite Darboux transform.

One step:

$$T = \mathbf{BA} + \tilde{\lambda}, \qquad A[y] = b(y' - wy),\ B[y] = \hat{b}(y' - \hat{w}y),$$

$$TP_n^{[0]} = \lambda_n P_n^{[0]}.$$

$$\hat{T} = \mathbf{AB} + \tilde{\lambda}$$

$$\hat{T}AP_n^{[0]} = \lambda_n AP_n^{[0]}$$

$$AP_n^{[0]} =: P_n^{[1]}$$

(s Darboux transforms: $A_s P_n^{[s-1]} =: P_n^{[s]}$)

Examples.(Gómez-Ullate, Marcellán, Milson JMAA (2013))

Jacobi differential operator:
$T[y] = (1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' = -n(n+\alpha+\beta+1)y$
with eigenfunctions $P_n^{[0]} = p_n^{(\alpha,\beta)}$.
$A[y] = (1-x)p_m^{(-\alpha,\beta)}y' + (m-\alpha)p_m^{(-\alpha-1,\beta-1)}y$, $B[y] = \frac{(1+x)y'+(1+\beta)y)}{p_m^{(-\alpha,\beta)}}$

$$T = BA - (m-\alpha)(m+\beta+1)$$

$$P_n^{[1]} = (1-x)p_m^{(-\alpha,\beta)}(p_n^{(\alpha,\beta)})' - (\alpha-m)p_m^{(-\alpha-1,\beta-1)}p_n^{(\alpha,\beta)}$$

$\{P_n^{[1]}\}$ OS on $[-1,1]$ w.r.t. $\frac{(1-x)^{\alpha-1}(1+x)^{\beta+1}}{(p_m^{(-\alpha,\beta)})^2}$.

Further properties.(Gómez-Ullate, Kasman, Kuijlaars, Milson, Miki, Odake, Sasaki, H. 2013-2019)

- $\deg P_n^{[s]} \geq n$. If $\deg P_n^{[s]} = n + m$, $P_n^{[s]}$ has n regular zeros in I, m exceptional zeros out of I.
- Recurrence relations:

(1) with variable dependent coefficients: ($L \geq 2$ depends on the number of Darboux transforms.)

$$P_n^{[L-1]} = \sum_{k=-L}^{L} r_{n,k}^{[L]}(x) P_{n+k}^{[L-1]}.$$

(2) with constant coefficients: ($L \geq 2$ depends on the codimension.)

$$Q_s P_n^{[s]} = \sum_{k=-L}^{L} u_{n,k} P_{n+k}^{[s]}.$$

Reminder to ordinary orthogonal polynomials

- If the recurrence coefficients $a_n \to a > 0$, $b_n \to b$, then

$$\nu_n \to \omega_{a,b}(x) = \frac{1}{\pi\sqrt{(b+2a-x)(x-b+2a)}} = \mu_e([b-2a, b+2a])$$

(Van Assche)

- supp$\mu \subset C(0,1)$ or supp$\mu \subset [-1,1]$ + Szegő condition, then

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Asymptotics of Recurrence Coefficients

One-step Darboux transform case.

$\{P_n^{[1]}\}_{n=0}^{\infty}$ ONS I w.r.t. $W := \frac{pw_0}{b^2}$, w_0=Laguerre/Hermite/Jacobi weight. Assume that b has simple zeros, $\frac{p}{b} = \frac{\tilde{p}}{\tilde{b}}$

$$Q(x) = \int^x \tilde{b}, \quad \deg Q = L,$$

$$QP_n^{[1]} = \sum_{k=-L}^{L} u_{n,k} P_{n+k}^{[1]}.$$

$u_{n,k}$ (Odake):

$BQP_n^{[1]} = \sum_{k=-L}^{L} a_{n,k} P_{n+k}^{[0]}, \quad \Rightarrow \quad u_{n,k} = \frac{a_{n,k}}{\lambda_{n+k} - \tilde{\lambda}}$

$$BQP_n^{[1]} = \tilde{p}b\left(P_n^{[0]}\right)' - \tilde{p}gP_n^{[0]} + (\lambda_n - \tilde{\lambda})QP_n^{[0]},$$

$$p\left(P_n^{[0]}\right)' = A_nP_{n+1}^{[0]} + B_nP_n^{[0]} + C_nP_{n-1}^{[0]}$$

$\tilde{b}(x) = \sum_{k=0}^{m} d_k x^k$, $(\tilde{p}g)(x) = \sum_{k=0}^{m+1} c_k x^k$

$$BQP_n^{[1]} = A_n \sum_{k=0}^{m} d_k \left(x^k P_{n+1}^{[0]}\right) + C_n \sum_{k=0}^{m} d_k \left(x^k P_{n-1}^{[0]}\right)$$

$$+ \left(B_n d_0 - c_0 + \sum_{k=1}^{m} \left(B_n d_k - c_k + \frac{\lambda_n - \tilde{\lambda}}{k} d_{k-1}\right) x^k + \right.$$

$$\left. \left((\lambda_n - \tilde{\lambda})\frac{d_m}{m+1} - c_{m+1}\right) x^{m+1}\right) P_n^{[0]}.$$

$$x^k P_n^{[0]} = \sum_{j=-k}^{k} (s_{k,|j|} + e_{k,n,j}) P_{n+j}^{[0]}, \ \lim_{n\to\infty} e_{k,n,j} = 0$$

$$s_{k,j} = \sum_{i=0}^{\left[\frac{k-|j|}{2}\right]} \binom{k}{|j|+2i} \binom{|j|+2i}{i} a^{|j|+2i} b^{k-|j|-2i}$$

X_m Jacobi polynomials

The limit of the recurrence coefficients: $a_n \to \frac{1}{2}, \quad b_n \to 0.$

$QP_n^{[1]} = \sum_{k=-L}^{L} u_{n,k} P_{n+k}^{[1]}, \; \tilde{b}(x) = \sum_{k=0}^{m} d_k x^k$

$\lim_{n\to\infty} u_{n,j} =: U_{|j|}$

Proposition. (H.)

$$U_{|j|} = \begin{cases} \sum_{p=\max\{l,1\}}^{\left[\frac{m+1}{2}\right]} \frac{d_{2p-1}}{2p} \binom{2p}{p-l} \frac{1}{2^{2p}}, & \text{if } |j| = 2l \\ \sum_{p=l}^{\left[\frac{m}{2}\right]} \frac{d_{2p}}{2p+1} \binom{2p+1}{p-l} \frac{1}{2^{2p+1}}, & \text{if } |j| = 2l+1. \end{cases}$$

Exceptional Jacobi Polynomials of Codimension 1

$\tilde{b}(x) = d_1 x + d_0$, $K_n(x,y) := \sum_{k=0}^{n-1} P_k^{[1]}(x) P_k^{[1]}(y)$,
$W(x) := W(\alpha,\beta)(x) = (1-x^2)\frac{(1-x)^\alpha(1+x)^\beta}{b^2(x)}$.

$$d\mu_N(x) = \frac{1}{N} K_N(x,x) W(x) dx \quad d\mu_e(x) = \omega(x)dx = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx$$

Theorem(H)

$$\lim_{N\to\infty} \mu_N = \mu_e$$

on $[-1,1]$, in weak-star sense.

Lemma(H) For all $k = 0, 1, \ldots$

$$\lim_{n\to\infty} \langle Q^k P_n^{[1]}, P_n^{[1]} \rangle_W = \int_{-1}^{1} Q^k(x)\omega(x)dx.$$

"Average Characteristic Polynomial"

$\{P_n\}_{n=0}^{\infty}$ ordinary or exceptional ONS, $K_N(x,y)$ as above, Q: $QP_n = \sum_{j=-L}^{L} u_{n,j} P_{n+j}$

$x_1, \ldots, x_N$: random variables, the joint probability distribution on $\mathbb{R}^N$:

$$\varrho_N(x_1, \ldots, x_N) = c(n,N) \det |K_n(x_i, x_j)|_{i,j=1}^{N} \prod_{i=1}^{N} W(x_i),$$

$c(n,N)$: normalization factor. Expectation $\mathbb{E}$ refers to ϱ.

Empirical distribution:

$$\widehat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}.$$

Let $f : I \longrightarrow \mathbb{R}$ (!)

$$\mathbb{E}\left(\sum_{i_1 \neq \cdots \neq i_k} f(x_{i_1}) \ldots f(x_{i_k}) \right)$$

$$= c(n,k) \int_{I^k} f(x_1) \ldots f(x_k) \det |K_n(x_i,x_j)|_{i,j=1}^{k} \prod_{i=1}^{k} d\mu(x_i)$$

Specially: $k = 1$ $n = N$

$$\mathbb{E}\left(\int f \widehat{\mu}_N \right) = \int_I f(x) \frac{1}{N} K_N(x,x) W(x) dx = \int_I f d\mu_N.$$

$\widehat{\mu}_N^Q = \frac{1}{N}\sum_{i=1}^N \delta_{Q(x_i)}$
The modified average characteristic polynomial:

$$\chi_N(z) := \chi_N(z)^Q(z) = \mathbb{E}\left(\prod_{i=1}^N (z - Q(x_i))\right).$$

z_i the zeros of $\chi_N(z)$
$\nu_N = \frac{1}{N}\sum_{i=1}^N \delta_{z_i}$.

Theorem.(H)
If there is a B such that $|u_{n,j}| \leq B$ for all n, j, then for all $l \geq 0$

$$\lim_{N\to\infty} \left|\mathbb{E}\left(\int x^l d\widehat{\mu}_N^Q(x)\right) - \int x^l d\nu_N(x)\right| = 0.$$

$\pi_N : f(x) \mapsto \int_I K_N(x,t) f(t) d\mu(t), \qquad M : f(x) \mapsto Q(x) f(x)$

Lemma.(H)

$$\mathbb{E}\left(\int x^l d\widehat{\mu}_N^Q(x)\right) = \frac{1}{N}\mathrm{Tr}(\pi_N M^l \pi_N),$$

$$\int x^l d\nu_N(x) = \frac{1}{N}\mathrm{Tr}\left((\pi_N M \pi_N)^l\right).$$

Let $z_i = Q(y_i)$, $\tilde{\nu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$

Corollary.(H)

In X_m Jacobi case for all $l \geq 0$

$$\lim_{N\to\infty}\left|\int_{-1}^{1} Q^l d\mu_N - \int Q^l d\tilde{\nu}_N\right| = 0.$$