Sampling, Marcinkiewicz-Zygmund Inequalities, Approximation, and Quadrature Rules

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Motivation

Given samples $f(x_j), j = 1, \ldots, m$, of a function $f \in C(M)$, $M \subseteq \mathbb{R}^d$ or M some manifold, what can you say about f?

How well can you approximate

- f from samples?
- $\int_M f(x) d\mu(x)$ from samples (Question of N. Trefethen)
- Optimal sampling strategies? (Cohen, Migliorati, Adcock, . . .)

Ingredients

- Increasing sequence of finite-dimensional subspaces ("polynomials")
- Sobolev spaces
- Marcinkiewicz-Zygmund families
- (Weighted) least squares problems
- Approximation theory

Simplicity

Model case: torus and trigonometric polynomials

- $M = \mathbb{T}$
- $\mathcal{T}_n$ trigonometric polynomials of degree n,
$$p(x) = \sum_{k=-n}^{n} c_k e^{2\pi ikx} \in \mathcal{T}_n.$$
- Sampling sets: $\mathcal{X}_n = \{x_{n,k} : k = 1, \dots, L_n\} \subseteq (-1/2, 1/2]$.
- Weights $\{\tau_{n,k} : k = 1, \dots, L_n\}$ associated to every $x_{n,k} \in \mathcal{X}$.

Definition

$\mathcal{X} = \{\mathcal{X}_n\}$ is called a Marcinkiewicz-Zygmund family, if there exist constants $A, B > 0$ such that

$$A\|p\|_2^2 \le \sum_{k=1}^{L_n} |p(x_{n,k})|^2 \tau_{n,k} \le B\|p\|_2^2 \qquad \textit{for all } p \in \mathcal{T}_n\,.$$

$\kappa = B/A$ global condition number of $\mathcal{X}$

Approximation from samples

Given $f \in C(\mathbb{T})$, solve least squares problems

$$p_n = \operatorname{argmin}_{p \in \mathcal{T}_n} \sum_{k=1}^{L_n} |f(x_{n,k}) - p(x_{n,k})|^2 \tau_{n,k} \, .$$

p_n best L^2-approximation of f in $\mathcal{T}_n$ from samples $\mathcal{X}_n$.

Problem: p_n versus f?

Orthogonal projection of f onto $\mathcal{T}_n$

$$P_n f(x) = \sum_{|k| \leq n} \hat{f}(k) e^{2\pi i k x}$$

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Smoothness and approximability

Sobolev spaces $H^\sigma(\mathbb{T})$ with norm

$$\|f\|_{H^\sigma} = \Big(\sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2 (1+k^2)^\sigma\Big)^{1/2} .$$

Lemma

(i) Sobolev embedding: $H^\sigma \hookrightarrow C(\mathbb{T})$ *for* $\sigma > 1/2$.
(ii) Convergence rate:

$$\|f - P_n f\|_2 \le \|f - P_n f\|_\infty \le C_\sigma \|f\|_{H^\sigma} \; n^{-\sigma+1/2} .$$

(iii) Sampling:

$$\sum_{k=1}^{L_n} |f(x_{n,k})|^2 \tau_{n,k} \le B\|f\|_\infty^2 \le BC_\sigma^2 \|f\|_{H^\sigma}^2 .$$

Approximation from samples

Theorem

Let $\mathcal{X}$ be a Marcinkiewicz-Zygmund family with associated weights τ and condition number $\kappa = B/A$.

(i) If $f \in H^\sigma$, then

$$\|f - p_n\|_2 \leq C_\sigma \sqrt{1+\kappa^2} \|f\|_{H^\sigma} n^{-\sigma+1/2} .$$

(ii) If $f \in C(\mathbb{T})$ extends to an analytic function on strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \rho_0\}$, then for every $\rho < \rho_0$

$$\|f - p_n\|_2 = \mathcal{O}(e^{-\rho n}) .$$

- $C_\sigma \approx (2\sigma - 1)^{-1/2}$

Quadrature rules

Sampling inequality leads to reconstruction formula [1]

$$p = \sum_{k=1}^{L_n} p(x_{n,k}) e_{n,k} \qquad \text{for all } p \in \mathcal{T}_n \, .$$

Set

$$w_{n,k} = \int_0^1 e_{n,k}(x)\, dx \, .$$

$$I_n(f) = \sum_{k=1}^{L_n} f(x_{n,k}) w_{n,k}$$

Then I_n is a quadrature rule and *exact on* $\mathcal{T}_n$

$$I(f) = \int_0^1 f(x)\, dx = I_n(f) \qquad \forall f \in \mathcal{T}_n \, .$$

[1] $\{e_{n,k} : k = 1, \dots, L_n\}$ is the dual frame of reproducing kernels

Quadrature rules

Theorem

$\mathcal{X}$ Marcinkiewicz-Zygmund family with condition number $\kappa = B/A$ and $\{I_n : n \in \mathbb{N}\}$ associated quadrature rules, $\sigma > 1/2$.
(i) If $f \in H^\sigma$ and $\sigma > 1/2$, then

$$|I(f) - I_n(f)| \leq (1 + \sqrt{\kappa}) C_\sigma \|f\|_{H^\sigma} n^{-\sigma+1/2} .$$

(ii) If f extends to analytic function on strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \rho_0\}$, then for $\rho < \rho_0$

$$|I(f) - I_n(f)| = \mathcal{O}(e^{-\rho n}) .$$

Approximation from samples

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(ii) If $f \in C(\mathbb{T})$ extends to an analytic function on strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \rho_0\}$, then

$$\|f - p_n\|_2 = \mathcal{O}(e^{-\rho' n}) .$$

- $C_\sigma \approx (2\sigma - 1)^{-1/2}$

Proof ideas

$p_n \in \mathcal{T}_n$ yields orthogonal decomposition

$$\|f - p_n\|_2^2 = \|\underbrace{f - P_n f}_{\in \mathcal{T}_n^{\perp}}\|_2^2 + \|\underbrace{P_n f - p_n}_{\in \mathcal{T}_n}\|_2^2 .$$

So $\|f - P_n f\|_2^2 = \mathcal{O}(n^{-2\sigma+1})$ for $f \in H^{\sigma}$.

Matrix form of least squares problem

Recall

$$p_n = \mathrm{argmin}_{p \in \mathcal{T}_n} \sum_{k=1}^{L_n} |f(x_{n,k}) - p(x_{n,k})|^2 \tau_{n,k} \qquad \in \mathcal{T}_n \,.$$

For $p \in \mathcal{T}_n$

$$p(x_k) = \sum_{l=-n}^{n} e^{2\pi i l x_k} \, \hat{f}(l)$$

Matrix form of least squares problem

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For $p \in \mathcal{T}_n$

$$\underbrace{p(x_k)}_{y} = \sum_{l=-n}^{n} \underbrace{e^{2\pi i l x_k}}_{U} \underbrace{\hat{f}(l)}_{\hat{f}}$$

Matrix form of least squares problem

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For $p \in \mathcal{T}_n$

$$\underbrace{\tau_{n,k}^{1/2} p(x_{n,k})}_{y_n} = \sum_{l=-n}^{n} \underbrace{\tau_{n,k}^{1/2} \, e^{2\pi i l x_{n,k}}}_{U_n} \underbrace{\hat{f}(l)}_{f_n}$$

Information about least squares problem

Lemma

For Marcinkiewicz-Zygmund family $\mathcal{X} = \{\mathcal{X}_n\}$
(i) spectrum of $T_n = U_n^* U_n$ is contained in the interval $[A, B]$ for all $n \in \mathbb{N}$, and
(ii) solution of least squares problem is $p_n = \sum_{|k| \leq n} a_{n,k} e^{2\pi ikx} \in \mathcal{T}_n$ with coefficients

$$a_n = (U_n^* U_n)^{-1} U_n^* y_n = U_n^\dagger y_n \, .$$

Proof of (i): $A \|p\|_2^2 \leq \sum |p(x_{n,k})|^2 \tau_{n,k} = \langle U_n f_n, U_n f_n \rangle$.

Main part of proof

Fourier coefficients

- of $P_n f$: $f_n = (\hat{f}(-n), \hat{f}(-n+1), \dots, \hat{f}(n-1), \hat{f}(n))$
- of p_n: $a_n = (U_n^* U_n)^{-1} U_n^* y_n$

Then

$$\begin{aligned} \|P_n f - p_n\|_2^2 &= \|f_n - a_n\|_2^2 \\ &= \|f_n - T_n^{-1} U_n^* y_n\|_2^2 \\ &= \|T_n^{-1}(T_n f_n - U_n^* y)\|_2^2 = \|T_n^{-1} U_n^* (U_n f_n - y)\|_2^2 \\ &\le A^{-2} B \|U_n f_n - y\|_2^2 , \end{aligned}$$

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$$(U_n f_n)_k = \tau_{n,k}^{1/2} \sum_{|l|\leq n} \hat{f}(l) e^{2\pi i l x_{n,k}} = \tau_{n,k}^{1/2} P_n f(x_{n,k}) \,,$$

and thus

$$\|U_n f_n - y\|_2^2 = \sum_{k=1}^{L_n} |P_n f(x_{n,k}) - f(x_{n,k})|^2 \tau_{n,k} \,,$$

Conclusion for $f \in H^\sigma$:

$$\begin{aligned}\|P_n f - p_n\|_2^2 &\leq A^{-2} B \sum_{k=1}^{L_n} |f(x_{n,k}) - P_n f(x_{n,k})|^2 \tau_{n,k} \\ &\leq A^{-2} B^2 \|f - P_n f\|_\infty^2 = \mathcal{O}(n^{-2\sigma+1})\,.\end{aligned}$$

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General result

Torus

ONB $e^{2\pi ikx}$ for $L^2(\mathbb{T})$

Fourier coefficients $\hat{f}(k)$

Eigenvalues k^2 of $-\frac{1}{4\pi^2}\Delta$

Sobolev space $H^\sigma(\mathbb{T})$

Trigonometric polynomials $\mathcal{T}_n$

Manifold M

ONB $\phi_k(x)$ for $L^2(M,\mu)$ [a]

Fourier coefficients $\langle f, \phi_k\rangle$

"Eigenvalues" $\lambda_k \geq 0$, $\lambda_k \to \infty$

Sobolev space $H^\sigma(M)$

"Polynomials" $\mathcal{P}_n$, $p = \sum_{k:\lambda_k \leq n^2} \hat{f}(k)\, \phi_k\}$

[a] with $\|\phi_k\|_\infty \leq C$

Norm of $H^\sigma(M)$ is $\|f\|_{H^\sigma}^2 = \sum_k |\hat{f}(k)|^2 (1+\lambda_k)^\sigma < \infty$.

Marcinkiewicz-Zygmund Families on M and approximation by polynomials

Recall: $\mathcal{X} = \{x_{n,k} : k = 1, \dots, L_n\}$ is Marcinkiewicz-Zygmund family on M, if

$$A\|p\|_2^2 \leq \sum_{k=1}^{L_n} |p(x_{n,k})|^2 \tau_{n,k} \leq B\|p\|_2^2 \qquad \textit{for all } p \in \mathcal{P}_n\,.$$

Least squares problems

$$p_n = \operatorname{argmin}_{p \in \mathcal{P}_n} \sum_{k=1}^{L_n} |f(x_{n,k}) - p(x_{n,k})|^2 \tau_{n,k}\,.$$

Approximation from samples on M

Remainder for error estimate:

$$\phi_\sigma(n) = \Big(\sum_{k:\lambda_k > n^2} (1+\lambda_k)^{-\sigma} \Big)^{1/2}.$$

Theorem

Assume that $\mathcal{X} = \{\mathcal{X}_n : n \in \mathbb{N}\}$ is a Marcinkiewicz-Zygmund family for M with condition number $\kappa = B/A$ and associated weights $T = \{\tau_{n,k}\}$.
If $f \in H^\sigma(M)$ for $\sigma > \sigma_{\mathrm{crit}}$, then

$$\|f - p_n\|_2 \leq \sqrt{1+\kappa^2}\, \|f\|_{H^\sigma}\, \phi_\sigma(n).$$

Proof is identical.

Existence of Marcinkiewicz-Zygmund families

A very early result for the torus (KG, ’92)

Theorem

Fix $0 < \delta < 1$, $x_{n,k}$ *be such that* $x_{n,k} < x_{n,k+1}$ *in* $(-1/2, 1/2]$ *and*

$$\max_{k=1,\dots,L_n} \left(x_{n,k+1} - x_{n,k}\right) \leq \frac{\delta}{2n} \qquad \forall n \in \mathbb{N}$$

$$\tau_{n,k} = (x_{n,k+1} - x_{n,k-1})/2$$

Then $(\mathcal{X}, \tau)$ *is a Marcinkiewicz-Zygmund family for* $C(\mathbb{T})$ *with constants* $A = (1-\delta)^2$ *and* $B = (1+\delta)^2$*.*

Characterization of Marcinkiewicz-Zygmund families for torus by Ortega-Cerdà and Saludes

Marcinkiewicz-Zygmund families for sufficiently large covering density

- Filbir-Mashkar on compact (Riemannian) manifolds
- Necessary density conditions for Marcinkiewicz-Zygmund families on compact (Riemannian) manifolds (Ortega-Cerdà-Pridhnani)
- Brandolini, Choirat, et al. on compact manifold with covering density
- De Marchi and Kroó for multivariate polynomials
- Random constructions of Marcinkiewicz-Zygmund families in statistical learning (Cohen-Migliorati, Adcock)
- ......

Summary (Recommendation)

For understanding of
approximation from samples
or quadrature rules
study and construct
Marcinkiewicz-Zygmund families

Thank you!