Uncertainty principles for eventually constant sign bandlimited functions

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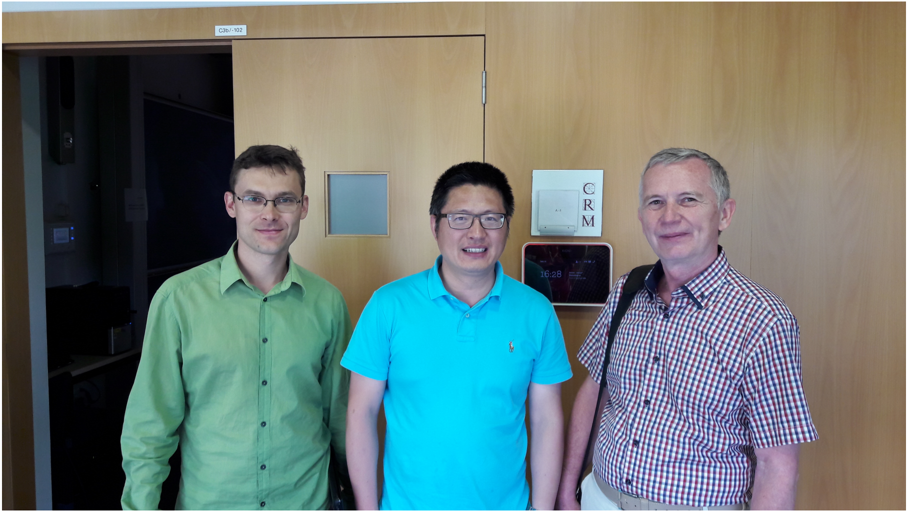


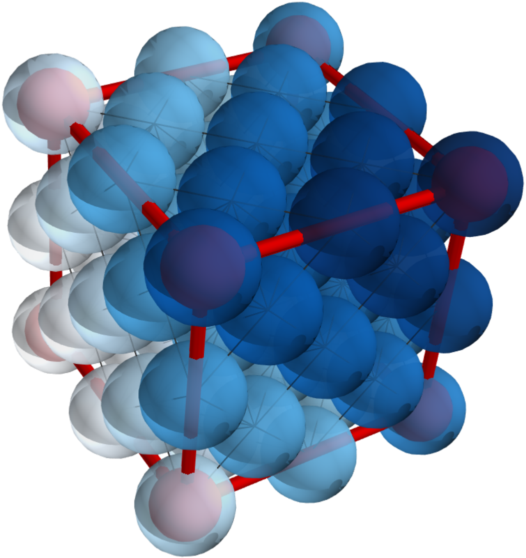
Sergey Tikhonov (left), Valerii Ivanov (right)

The results can be found on arXiv:1904.11328 [math.CA]

Some notation

$f\colon \mathbb{R}^n \to \mathbb{R}$	even continuous
$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x,y\rangle}\, dx$	the Fourier transform
$f(x) = f(\lvert x\rvert)$	radial functions
$\widehat{f}(\lvert y\rvert) = C_n \lvert y\rvert^{1-\frac{n}{2}} \cdot \int_0^\infty f(t) J_{\frac{n}{2}-1}(2\pi \lvert y\rvert t) t^{\frac{n}{2}}\, dt$	the Fourier transform of $f(\lvert x\rvert)$
J_α	the Bessel function
$q_{\alpha,1} < q_{\alpha,2} < \dots$	positive zeros of J_α
compact $\operatorname{supp} \widehat{f}$	bandlimited functions
$\widehat{f} \geq 0$ (distributional)	positive definite functions
rapidly decreasing smooth f	Schwartz functions
$B(x,r)$	the Euclidean ball

Sphere packing problem



The center sphere packing density

$$\delta_n = \limsup_{R\to\infty} \frac{N_R}{(2R)^n},$$

where

$$N_R = \max\left\{N\colon \quad \exists\,\{x_i\}_1^N \subset [-R,R]^n, \quad |x_i - x_j| \ge 2,\ i \ne j\right\}.$$

$\delta_n = \delta_n(\mathcal{P}_*)$, $\mathcal{P}_*$ the densest packing

Linear programming (Delsarte) bound

Let $R \gg 1$ and f be a positive definite Schwartz functions s.t. $\widehat{f}(0) = 1$ and $f \leq 0$ outside of $B(0,2)$ (Fig. 1).

Consider

$$f_{\mathsf{per}}(x) = \sum_{\nu \in \mathbb{Z}^n} f(x + 2R\nu) = \frac{1}{(2R)^n} \sum_{\mu \in (2R)^{-1}\mathbb{Z}^n} \widehat{f}\,(\mu) e^{2\pi i \langle x, \mu \rangle}.$$

Then

$$\sum_{i,j=1}^{N_R} f_{\mathsf{per}}(x_i - x_j) \begin{cases} \overset{f|_{|x|\geq 2} \leq 0}{\leq} \displaystyle\sum_{i=j=1}^{N_R} f_{\mathsf{per}}(x_i - x_j) = N_R f_{\mathsf{per}}(0) \overset{f|_{|x|\geq 2} \leq 0}{\leq} N_R f(0) \\ = \displaystyle\frac{1}{(2R)^n} \sum_{\mu \in (2R)^{-1}\mathbb{Z}^n} \widehat{f}\,(\mu) \left| \sum_{j=1}^{N_R} e^{2\pi i \langle x_j, \mu \rangle} \right|^2 \overset{\widehat{f} \geq 0,\ \widehat{f}(0)=1}{\geq} \frac{N_R^2}{(2R)^n} \end{cases}.$$

DG (2000, bandlimited functions), H. Cohn and N. Elkies (2001).

Thus,

$$\delta_n \leq \delta_n^{\mathsf{LP}} := \inf \left\{ f(0) : \quad \widehat{f} \geq 0, \quad \widehat{f}(0) = 1, \quad f\big|_{\mathbb{R}^n \setminus B(0,2)} \leq 0 \right\},$$

where one can assume $f, \widehat{f} \in L^1(\mathbb{R}^n)$.

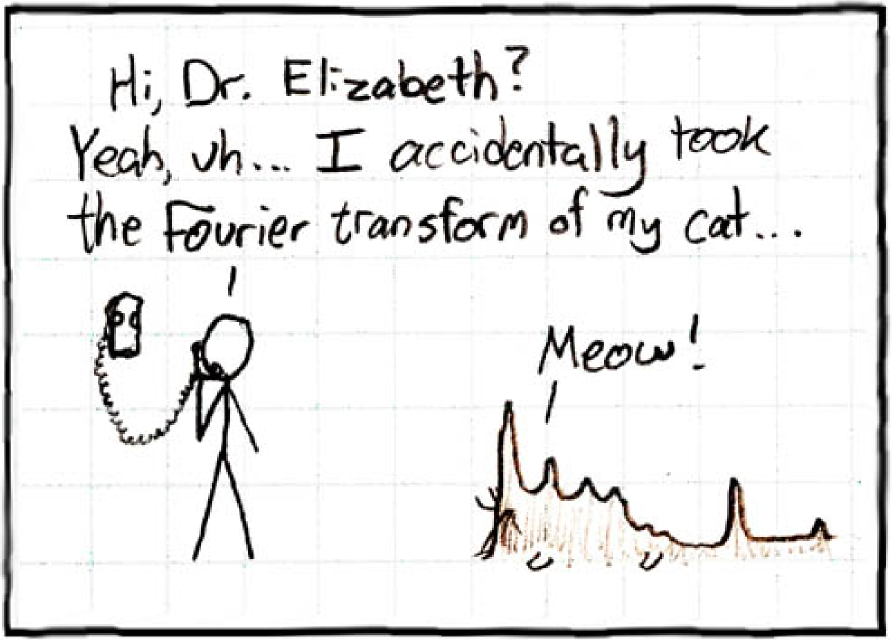
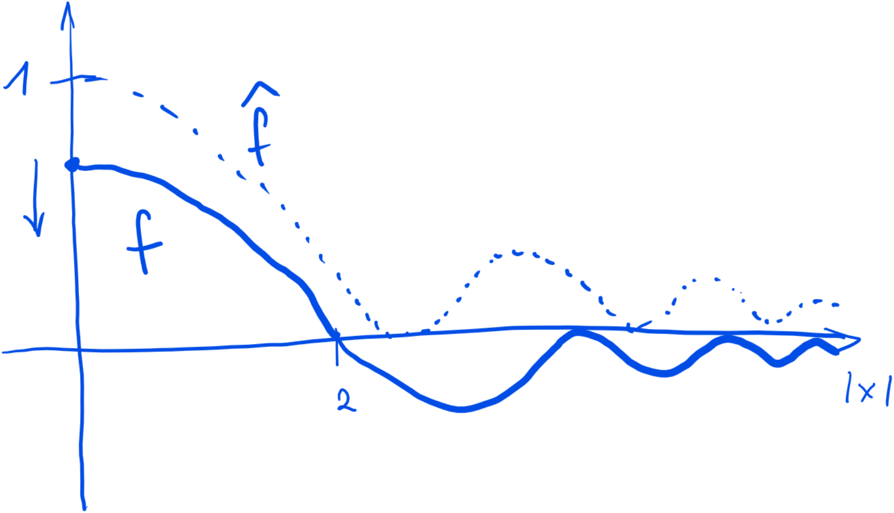

Hi, Dr. Elizabeth?
Yeah, uh... I accidentally took the Fourier transform of my cat...
Meow!

Fig. 1: δ_n^{LP}



Known results

$\delta_1 = \delta_1^{\mathsf{LP}}$	$\mathbb{Z}$	
δ_2	A_2	L. Fejes Tóth (1953) ($?= \delta_2^{\mathsf{LP}}$)
δ_3	fcc	T. Hales (1998–2006, Kepler conjecture)
$\delta_8 = \delta_8^{\mathsf{LP}}$	E_8	M. Viazovska (2016, via modular forms)
$\delta_{24} = \delta_{24}^{\mathsf{LP}}$	Λ_{24}	H. Cohn, A. Kumar, S. Miller, D. Radchenko, M. Viazovska (2016)

The Turan problem

Note that the problem

$$\inf\left\{f(0)\colon\quad \widehat{f}\geq 0,\quad \widehat{f}(0)=1,\quad f\big|_{\mathbb{R}^n\setminus B(0,2)}\overset{\not\leq}{=}0\right\}$$

now is known as the Turan problem (Fig. 2).

The function $f_*(x)=\frac{\chi_{B(0,1)}*\chi_{B(0,1)}}{\mathrm{vol}\,(\chi_{B(0,1)})^2}$ is extremal.

C. Siegel (1935, Minkowski's theorem),

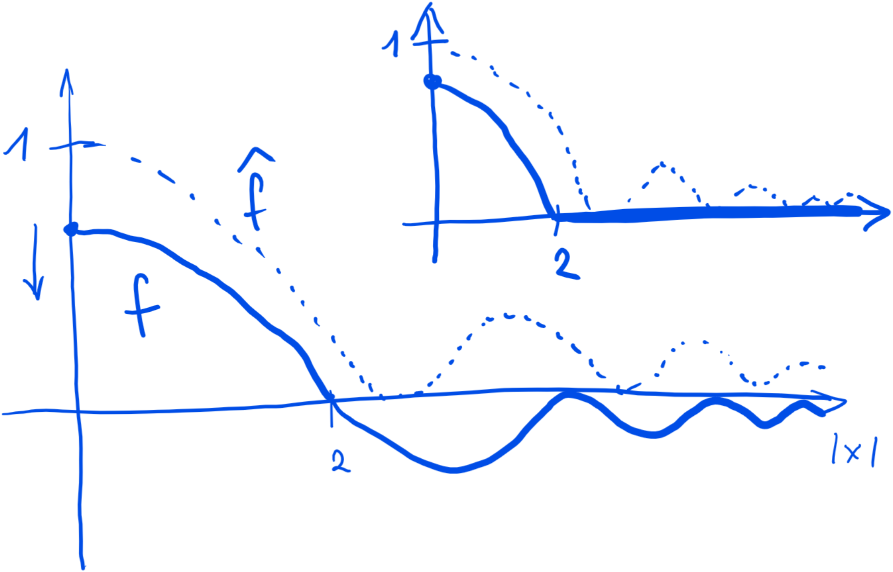
J. Holt, J. Vaaler (1996, one-sided approximation), DG (2001),
M. Kolountzakis, Sz. Révész (2003), V. Arestov, E. Berdysheva (2001, polytops),

DG, V. Ivanov, A. Manoshina, Yu. Rudomazina ($d=1$),

G. Bianchi, M. Kelly (2014, the Blaschke–Santalo inequality and the Mahler problem),

E. Hlawka (1981), DG, C. Тихонов (2018, the Wiener problem).

Fig. 2: Turan problem



Uncertainty principles of Bourgain, Clozel, and Kahane

J. Bourgain, L. Clozel, and J.-P. Kahane (2010) [BCK10],

F. Gonçalves, D. Oliveira e Silva, and S. Steinerberger (2017) [GOS17],

H. Cohn and F. Gonçalves (2017) [CG17].

Let $f, \widehat{f} \in L^1(\mathbb{R}^n) \setminus \{0\}$. Find

$$\sup\{|x|\colon f(x) < 0\} \cdot \sup\{|x|\colon \widehat{f}(x) < 0\} \to \inf_{f(0)\le 0,\ \widehat{f}(0)\le 0} =: A_n^+$$

([BCK10])

$$\sup\{|x|\colon f(x) < 0\} \cdot \sup\{|x|\colon \widehat{f}(x) > 0\} \to \inf_{f(0)\ge 0,\ \widehat{f}(0)\le 0} =: A_n^-$$

([CG17])

(Fig. 3, 4).

Fig. 3: A_n^+

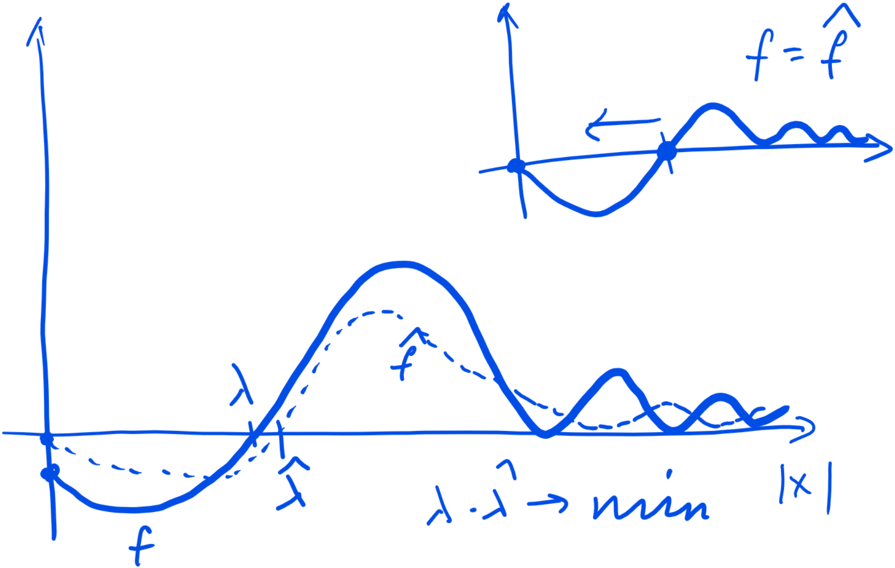
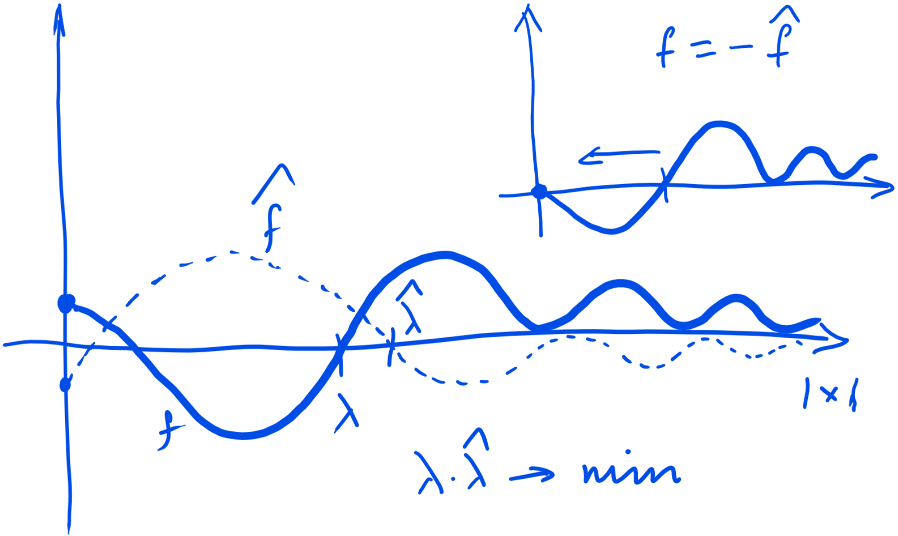


Fig. 4: A_n^-



Known results

There exists the radial extremal functions f s.t. $f(0) = \widehat{f}(0) = 0$ and $\widehat{f} = \pm f \quad (A_n^{\pm})$	[GOS17], [CG17]
$\frac{n}{2\pi e} < A_n^+ < \frac{n+2}{2\pi}$	[BCK10]
$0.2025 \leq A_1^+ \leq 0.353$	[GOS17]
$A_{12}^+ = 2$	[CG17]
$A_n^- \leq 4\,(\delta_n^{\mathsf{LP}})^{2/n}$	[CG17]
$\frac{n}{2\pi e} \leq A_n^- \leq \frac{0.6409...n(1+o(1))}{2\pi}$	[CG17] (using Kabatiansky–Levenshtein bound)
$A_1^- = 1,\ A_8^- = 2,\ A_{24}^- = 4$	via δ_n^{LP}

One-sided approximation by positive definite functions

H. Cohn and M. de Courcy-Ireland (2018, discrete energy of a packing $\mathcal{P}$):

$$\rho\widehat{f}(0) - f(0) \to \inf, \quad \widehat{f} \geq 0, \quad f(x) \leq \psi(|x|).$$

The potential $\psi(\sqrt{r})$ completely monotonic and $\rho > 0$ is the density of $\mathcal{P}$.

They considered the following radial function:

$$\frac{|x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x|)}{(1-\frac{|x|^2}{q_1^2})\dots(1-\frac{|x|^2}{q_{m+1}^2})} \qquad (q_k \text{ are positive zeros of } J_{\frac{n}{2}-1}),$$

and proved that this function is positive definite.

Note that $\psi = \chi_{[0,2]} \implies \delta_n^{\mathsf{LP}}$.

M. Gaál and Sz. Gy. Révész (2019):

$$C_n = \inf C, \quad f \leq C\chi_{[0,1]} - \chi_{[0,2]} \implies \int_{B(0,2)} \varphi \leq C_n \int_{B(0,1)} \varphi,$$

where $\varphi, \widehat{\varphi} \geq 0$.

Our main purpose has been complete solving of the generalized $A_n^{\pm}$-type problems for *bandlimited functions*.

Some historical background: Trigonometric polynomials

V. Arnold (1996): Let

$$t_n = \sum_{1 \le |k| \le n-1} c_k e^{2\pi k x}, \quad t_n(x) \le 0, \quad x \notin [a,b] \implies b - a \ge \frac{1}{n}.$$

A. Babenko (1984):

$$\inf \operatorname{mes} \{x \in \mathbb{T} \colon t_n \ge 0\} = \frac{1}{n}.$$

V. Yudin (2002): $t_{k,n} = \sum_{k \le |j| \le n-k} c_j e^{2\pi j x}$,

$$t^*_{n,k}(x) = \frac{(\cos \pi n x)^2}{(\cos 2\pi x - \cos \frac{1}{2n}) \dots (\cos 2\pi x - \cos \frac{2k-1}{2n})}.$$

Logan's problems

Problem 0: *Find the smallest* $\lambda_0 > 0$ *s.t. that* $f(x) \leq 0, \;\; x > \lambda_0$, *where*

$$f(x) = \int_0^1 \cos(2\pi x t)\, d\mu(t), \quad d\mu \geq 0, \quad f(0) = 1.$$

Logan showed that admissible functions in Problem 0 are integrable, $\lambda_0 = 1/2$, and the unique extremizer is $f_0(x) = \frac{(\cos(\pi x))^2}{1-(2x)^2}$ satisfying $\int_{\mathbb{R}} f(x)\, dx = 0$.

Problem 1: *Find the smallest* $\lambda_1 > 0$ *such that* $f(x) \geq 0, \;\; x > \lambda_1$, *where* f *is a integrable function from Problem 0 s.t.* $\int_{\mathbb{R}} f(x)\, dx = 0$.

It turns out that admissible functions are integrable with respect to the weight x^2, and $\lambda_1 = 3/2$. Moreover, the unique extremizer is $f_1(x) = \frac{(\cos(\pi x))^2}{(1-(2x)^2)(1-(2x/3)^2)}$ satisfying $\int_{\mathbb{R}} x^2 f(x)\, dx = 0$.

It is natural to continue and consider the following problems.

Logan m-problem

Let $m = 0, 1, 2, \ldots$. Find

$$\sup\{|x|\colon (-1)^m f(x) > 0\} \cdot \sup\{|x|\colon x \in \operatorname{supp} \widehat{f}\,\} \to \inf =: A_n^{\mathsf{band}}(m),$$

where the infimum is taken over all nontrivial *positive definite bandlimited* functions f s.t. if $m \geq 1$, then $f \in L^1(\mathbb{R}^d, |x|^{2m-2}\, dx)$ and

$$\int_{\mathbb{R}^d} |x|^{2k} f(x)\, dx = 0 \quad \text{for} \quad k = 0, \ldots, m-1.$$

Logan (m, s)-problems

Let $m, s = 0, 1, 2, \ldots$. Find

$$\sup\{|x| \colon (-1)^m f(x) > 0\} \cdot \sup\{|x| \colon x \in \operatorname{supp} \widehat{f}\,\} \to \inf =: A_n^{\text{band}}(m, s),$$

where the infimum is taken over all nontrivial *even bandlimited* functions $f \in L^1(\mathbb{R}^d, |x|^{2m}\,dx)$ s.t.

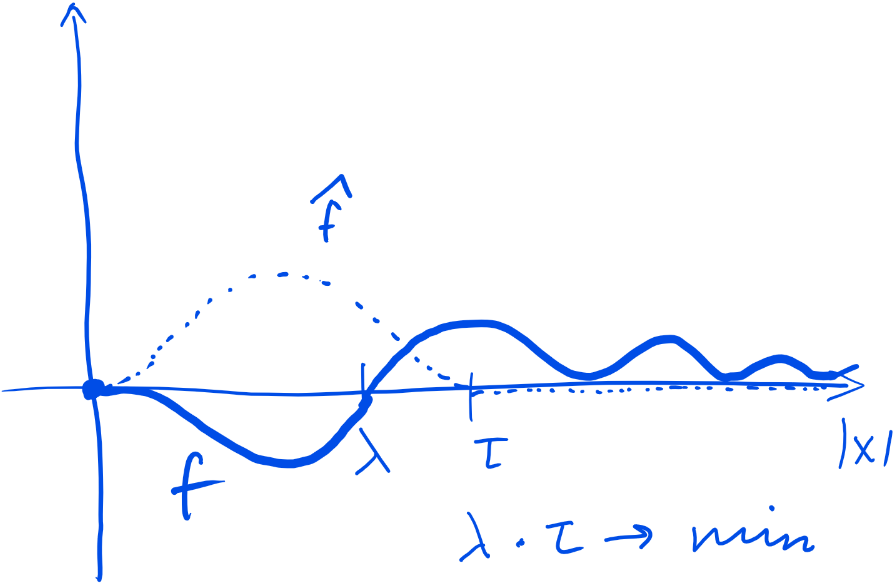
$$\begin{cases} \int_{\mathbb{R}^d} |x|^{2k} f(x)\,dx = 0, & k = 0, \ldots, m-1, \\ \int_{\mathbb{R}^d} |x|^{2\ell}\,\widehat{f}\,(x)\,dx = 0, & \ell = 0, \ldots, s-1, \end{cases}$$

and

$$\int_{\mathbb{R}^d} |x|^{2m} f(x)\,dx \geq 0, \qquad \int_{\mathbb{R}^d} |x|^{2s} \widehat{f}\,(x)\,dx \leq 0$$

(Fig. 5).

Fig. 5: Problem $A_n^{\text{band}}(m, s)$



Interrelation between $A_n^{\pm}$ and $A_n^{\mathsf{band}}(0,0)$

If $m = s = 0$, then

$$A_n^{\mathsf{band}}(0,0) = \inf_{f(0)\le 0,\ \widehat{f}(0)\ge 0} \sup\{|x| \colon f(x) > 0\} \cdot \sup\{|x| \colon x \in \operatorname{supp} \widehat{f}\,\}.$$

Therefore, $A_n^{\pm} \le A_n^{\mathsf{band}}(0,0)$.

Theorem 1

One has $A_n^{\mathsf{band}}(m) = \frac{q_{m+1}}{\pi}$*, where* q_k *denotes* k*-th positive zero of* $J_{\frac{n}{2}-1}$*.*

The function $\frac{\left(|x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x|)\right)^2}{\left(1-\frac{|x|^2}{q_1^2}\right)\dots\left(1-\frac{|x|^2}{q_{m+1}^2}\right)}$ *is the unique extremizer up to a positive constant.*

Moreover, this function satisfies $\int_{\mathbb{R}^d} |x|^{2m} f(x)\, dx = 0$*.*

In the case $m = 0,\, 1$ this theorem was proved by DG (2000).

Theorem 2

One has $A_n^{\text{band}}(m,s) = \frac{q_{\frac{n}{2}+s,m+1}}{\pi}$, $\qquad J_\alpha(q_{\alpha,k}) = 0, \quad \alpha = \frac{n}{2} + s$.

Each extremizer $f(x)$ *has the form* $r(x) \cdot f_{\alpha,m}(|x|)$, *where*

$$f_{\alpha,m}(t) = \frac{\left(|x|^{-\alpha} J_\alpha(|x|)\right)^2}{\left(1 - \frac{|x|^2}{q_{\alpha,1}^2}\right) \dots \left(1 - \frac{|x|^2}{q_{\alpha,m+1}^2}\right)}$$

and

$$r(x) = \sum_{j=0}^{s+1} |x|^{2s+2-2j} h_{2j}(x) \geq 0, \quad |x| \geq q_{\alpha,m+1},$$

$h_{2j}(x)$ *are even harmonic polynomials of order at most* $2j$ *s.t.* $h_0(0) > 0$, $h_2(0) = \ldots = h_{2s+2}(0) = 0$.

Moreover, $\int_{\mathbb{R}^d} |x|^{2m} f(x)\, dx = \int_{\mathbb{R}^d} |x|^{2s} \widehat{f}(x)\, dx = 0$.

This theorem again implies that $A_n^+ \leq \frac{q_{\frac{n}{2},1}}{\pi} = \frac{n+O(n^{1/3})}{2\pi}$ as $n \to \infty$.

Idea 1 of proof. Extremal quadrature formulas

The case $\delta_8^{\mathbf{LP}}$: Consider the Korkine–Zolotareff lattice E_8. Using self-duality of E_8 and the Poisson summation formula with a radial Schwartz function f, we obtain

$$\sum_{k=0}^{\infty} N_k f(\sqrt{2k}) = \frac{1}{\sqrt{\det E_8}} \sum_{k=0}^{\infty} N_k \widehat{f}(\sqrt{2k}),$$

where positive integers N_k associated with modular forms. Thus,

$$f(0) \geq (\det E_8)^{-1/2} \widehat{f}(0) \quad \text{if} \quad \widehat{f} \geq 0 \quad \text{and} \quad f(t) \leq 0 \quad \text{for} \quad t \geq \sqrt{2}.$$

Construction of the extremal functions needs the theory of modular forms.

Our case: We use

$$\int_0^{\infty} f(t) t^{2\alpha+1}\, dt = \sum_{l=0}^{r-1} \alpha_{l,r} f^{(2l)}(0) + \sum_{k=1}^{\infty} \gamma_{k,r} f\Big(\frac{2q_{\alpha+r,k}}{\tau}\Big), \quad \gamma_{k,r} > 0.$$

R. Ghanem and C. Frappier (1998).

Idea 2 of proof. Positive definiteness of the extremizer. A

Let $g_{\alpha,m}(|x|) = \frac{|x|^{-\alpha} J_\alpha(|x|)}{\left(1-\frac{|x|^2}{q_{\alpha,1}^2}\right)\ldots\left(1-\frac{|x|^2}{q_{\alpha,m+1}^2}\right)}$.

H. Cohn, M. de Courcy-Ireland (2018) proved positive definiteness of $g_{\alpha,m}$ in the case $\alpha = \frac{n}{2} - 1$ using

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\alpha)}\left(1 - \frac{t^2}{2n^2} + o(n^{-2})\right)}{P_n^{(\alpha,\alpha)}(1)} = C_\alpha t^{-\alpha} J_\alpha(t)$$

and the fact that $\frac{P_n^{(\alpha,\alpha)}(z)}{(z-r_{1,n})\ldots(z-r_{k,n})}$ is a linear combination of $P_0^{(\alpha,\alpha)}(z), \ldots, P_{n-k}^{(\alpha,\alpha)}(z)$ with nonnegative coefficients for each $k \le n$, where $r_{1,n} > \ldots > r_{n,n}$ are zeros of the Jacobi polynomial $P_n^{(\alpha,\alpha)}(z)$

The case $k = 1$ corresponds to Christoffel—Darboux formula, the case $k = 2$ was given by DG and V. Ivanov (2000), and the case $k \ge 1$ by H. Cohn and A. Kumar (2007).

We extend the Cohn–Kumar approach involving the Bessel translation operator.

Positive definiteness of $g_{\alpha,m}$. B

We found a direct way to prove positive definiteness of $g_{\alpha,m}$ using Sturm's theorem on zeros of linear combinations of eigenfunctions of Sturm–Liouville problem.

First, we show that the Fourier transform of $g_{\alpha,m}$ can be expressed via a linear combination of Bessel functions $\{t^{-\alpha}J_\alpha(q_k t)\}_{k=1}^{\infty} =: \Phi, \;\; t \in [0,1]$. Moreover, this combination has a zero of sufficiently large order at $t = 1$.

Then we prove the fact (probably new) that the system Φ forms *Chebyshev systems* on $[0,1)$. It means that any nontrivial linear combination $P(t) = \sum_{k=1}^{n} A_k \varphi_k(t)$ has at most $n-1$ zeros (counting multiplicity) on $[0,1)$.

To prove this we use the following deep Sturm's theorem.

Sturm's theorem (Sturm, 1836; Liouville, 1836)

Let $\{V_k\}_{k=1}^{\infty}$ *be the system of eigenfunctions associated to eigenvalues* $\rho_1 < \rho_2 < \dots$ *of the following Sturm–Liouville problem on* $[a, b]$*:*

$$(KV')' + (\rho G - L)V = 0,$$

$$(KV' - hV)(a) = 0, \quad (KV' + HV)(b) = 0,$$

where $G, K, L \in C[a, b]$*,* $K \in C^1(a, b)$*,* $K, G > 0$ *on* (a, b)*,* $h, H \in [0, \infty]$ *and* ρ *denotes the spectral parameter.*

Then for any nontrivial real polynomial of the form

$$P = \sum_{k=m}^{n} A_k V_k, \quad m, n \in \mathbb{N}, \quad m \leq n,$$

we have

$$m - 1 \leq |\{t \in (a, b) \colon P(t) = 0\}| \leq n - 1.$$

In particular, every k*-th eigenfunction* V_k *has exactly* $k - 1$ *simple zeros in* (a, b)*.*

This theorem is well known for trigonometric system as the Sturm–Hurwitz theorem.

P. Bérard and B. Helffer (2017).

Thank you for your attention!
Questions?