The Zeros of a Class of Dirichlet Functions

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Abstract:
We study a class of Dirichlet functions obtained as analytic continuation across the line of convergence of Dirichlet series which can be written as Euler products.
By using the geometric properties of the mapping realized by these functions, we tackle the problem of the multiplicity of their zeros.

1. Introduction

We are dealing with normalized series of the form :

$$\zeta_{A,\Lambda}(s) = \Sigma_{n=1}^{\infty} a_n e^{-\lambda_n s} = 1 + a_2 e^{-\lambda_2 s} + \ldots \qquad (1)$$

where $A = \{a_1 = 1, a_2, \ldots\}$ is an arbitrary sequence of complex numbers (the *coefficients* of the series) and $\Lambda = \{\lambda_1 = 0, \lambda_2, \ldots\}$ is a sequence of non decreasing real numbers (the *exponents* of the series).

When $\lambda_n = \ln n$, then the series has the form $\zeta_{A,\Lambda}(s) = \Sigma_{n=1}^{\infty} \frac{a_n}{n^s}$ and it is called *ordinary Dirichlet series.*

We suppose that the function $a(n) = a_n$ is totally multiplicative, i. e. $a(mn) = a(m)a(n)$ for every $m, n \in \mathbb{N}$.

Then for every prime decomposition $n = n_1^{k_1} n_2^{k_2} \ldots n_j^{k_j}$ we have $a(n) = a(n_1)^{k_1} a(n_2)^{k_2} \ldots a(n_j)^{k_j}$.

If the series is an ordinary one, then

$$\lambda(n) = \lambda_n = \ln n = k_1 \ln n_1 + k_2 \ln n_2 + \ldots + k_j \ln n_j = k_1 \lambda(n_1) + k_2 \lambda(n_2) + \ldots + k_j \lambda(n_j).$$

Assuming that we have also this last property for a general Dirichlet series, then it can be easily shown that the series appears also as an Euler product:

$$\zeta_{A,\Lambda}(s) = \Pi_{p \in \wp} (1 - a_p e^{-\lambda_p s})^{-1} \qquad (2)$$

where $\wp$ is the set of prime numbers. For such a series we have

$$\frac{\zeta'_{A,\Lambda}(s)}{\zeta_{A,\Lambda}(s)} = -\Sigma_{p \in \wp} \lambda_p a_p e^{-\lambda_p s} / (1 - a_p e^{-\lambda_p s}) \qquad (3)$$

The series on the left hand side of this equality have the same half plane of convergence and it must coincide with that of the right hand side.

We denote by e^{Λ} the sequence $\{e^{\lambda_1}, e^{\lambda_2}, \ldots\}$.

It is known that (see [2]) if $\zeta_{A,\Lambda}(s)$ has a finite abscissa of convergence, then the abscissa of convergence of $\zeta_{A,e^{\Lambda}}(s)$ is zero.

Moreover, if $\zeta_{A,e^{\Lambda}}(s)$ has a discrete set of singular points on the imaginary axis, then $\zeta_{A,\Lambda}(s)$ can be continued as a meromorphic function in the whole complex plane. Dirichlet L-series can be continued in this way.

Suppose that the series $\zeta_{A,\Lambda}(s)$ admits such a continuation and satisfies (2). We will call the extended function *Euler product function.*

Meromorphic continuations are possible also for the series $\zeta'_{A,\Lambda}(s)$ and for the right hand side term in (3).

Let us denote by $\varphi(s)$ the meromorphic function obtained on the right hand side in (3) by this continuation.

By the uniqueness theorem of analytic functions, $\varphi(s)$ coincides with $\frac{\zeta'_{A,\Lambda}(s)}{\zeta_{A,\Lambda}(s)}$, therefore it can have only simple poles.

The zeros of any order of $\zeta_{A,\Lambda}(s)$ are simple poles of $\varphi(s)$.

2. The Geometry of the mappings by General Dirichlet Series

From (1) it can be easily seen that $\lim_{\sigma\to+\infty} \zeta_{A,\Lambda}(\sigma + it) = 1$. We have shown in [3] that this happens uniformly with respect to t.

This simple fact has important consequences regarding the landscape of the pre-image of the real axis by a Dirichlet function (see Fig 1 below).

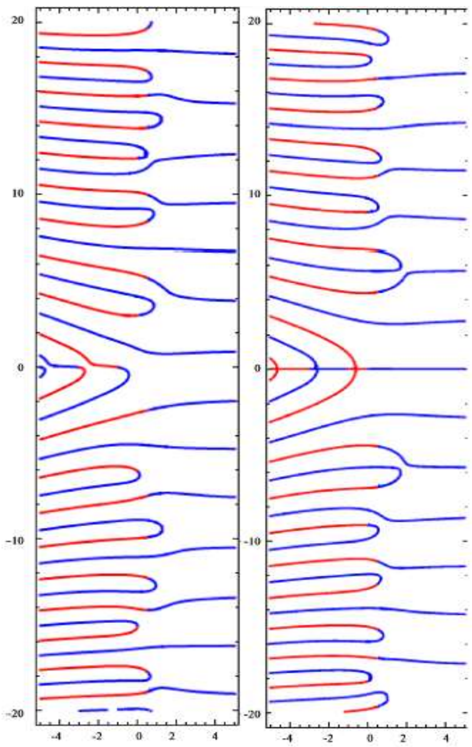

Fig. 1 The landscape of the pre-image of the real axis by Dirichlet functions

Let us list a few facts valid for any Dirichlet function:

a) The pre-image of the real axis by $\zeta_{A,\Lambda}(s)$ contains infinitely many disjoint curves Γ'_k extending for σ from $-\infty$ to $+\infty$ which are mapped each one bijectively by $\zeta_{A,\Lambda}(s)$ onto the interval $(1,+\infty)$ of the real axis. Consecutive curves Γ'_k and Γ'_{k+1} form infinite strips $S_k,\ k \in \mathbb{Z}$, where S_0 contains the real axis.

b) Every strip $S_k,\ k \neq 0$ contains a unique component $\Gamma_{k,0}$ of the pre-image of the real axis which is mapped bijectively by $\zeta_{A,\Lambda}(s)$ onto the interval $(-\infty, 1)$ and a unique unbounded component of the pre-image of the unit circle. It contains also a finite number of curves $\Gamma_{k,j},\ j \neq 0$ which are mapped bijectively onto the whole real axis. Every curve $\Gamma_{k,j}$ contains a unique zero of $\zeta_{A,\Lambda}(s)$.

c) The curves $\Gamma_{k,j}$ are disjoint, except that $\Gamma_{k,0}$ can meet $\Gamma_{k,1}$ or $\Gamma_{k,-1}$ into a double zero of $\zeta_{A,\Lambda}(s)$.

d) If we color, for example, red the pre-image of the negative real half axis and blue the pre-image of the positive real half axis, then the pre-image of any circle centered at the origin will meet alternatively the color red and the color blue. This is the *color alternating rule*. It is illustrated in Fig.2 above. The same rule applies also to the pre-image of the real axis by any derivative of $\zeta_{A,\Lambda}(s)$.

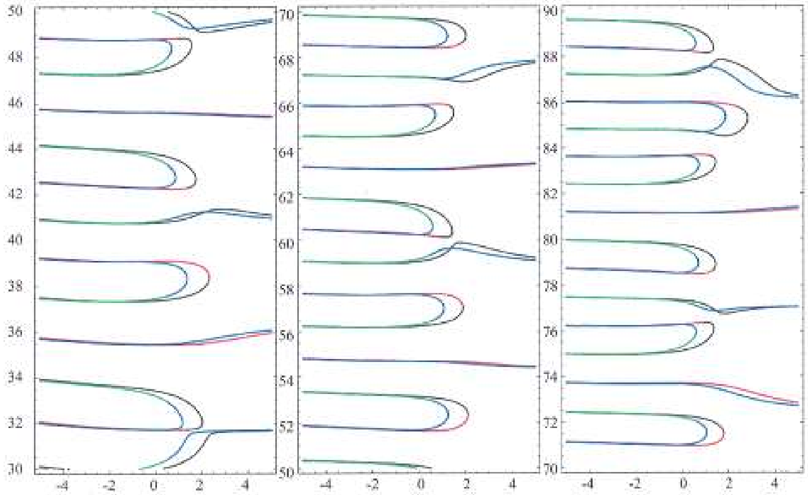

Fig.2- A sample of intertwining curves.

e) When represented in the same plane, the pre-image of the real axis by both $\zeta_{A,\Lambda}(s)$ and $\zeta'_{A,\Lambda}(s)$, come in couples of curves Γ'_k and Υ'_k, respectively $\Gamma_{k,j}$ and $\Upsilon_{k,j}$ which intersect two by two (the *intertwining curves*).

f) The intertwining curves intersect each other in points where the tangent to $\Gamma_{k,j}$, respectively Γ'_k is horizontal, or in multiple zeros, where the tangents do not exist. An illustration of this fact can be seen in Fig.2 above.

g) If we denote by **a** the color of the pre-image of the negative real half axis by $\zeta_{A,\Lambda}(s)$ and **b** that of the positive real half axis and by **c** and **d** the colors of the pre-image of the same half axes by $\zeta'_{A,\Lambda}(s)$, then color **a** can meet only color **d** and color **b** can meet only color **c,** except for the case of $\Gamma_{k,0}$ and $\Upsilon_{k,0}$, where color **d** meets both color **a** and **b**. This is the *color matching rule.*

h) The color matching rule forbids double zeros at the intersection of $\Gamma_{k,j}$ and $\Upsilon_{k,j}$ when $j \neq 0$.

i) If $s_{k,j}$ are the zeros of $\zeta'_{A,\Lambda}(s)$ in S_k, then the pre-image of the segment $l_{k,j}$ from $z = 1$ to $z = \zeta_{A,\Lambda}(s_{k,j})$ and the curves $\Gamma_{k,j}$ and Γ'_k bound fundamental domains which are mapped conformally by $\zeta_{A,\Lambda}(s)$ onto the whole complex plane with slits alongside these segments and the interval $[1,+\infty)$ of the real axis. In a similar way fundamental domains are obtained also for $\zeta'_{A,\Lambda}(s)$.

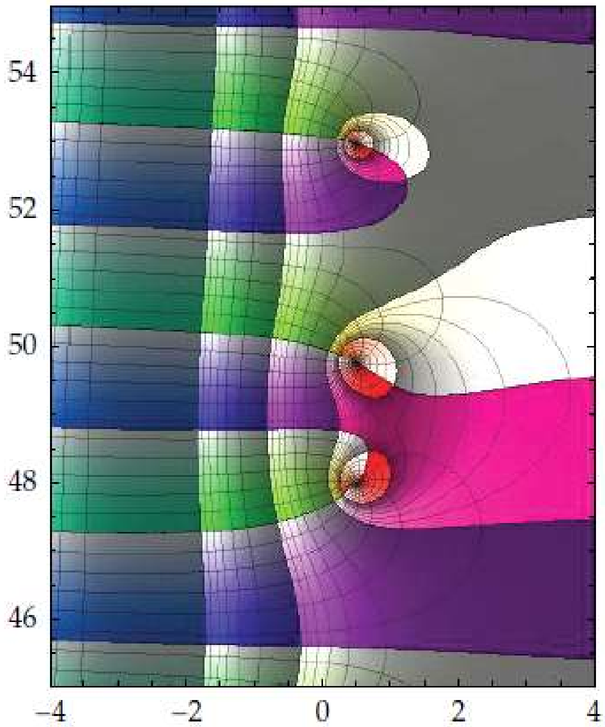

Fig 3. An illustration of the color alternating rule

3. Local mapping properties of analytic functions

It is known (see [1]) that if z_0 is a regular point of the analytic function $f(z)$, then in a neighborhood of z_0 we have:

$$f(z) = f(z_0) + (z - z_0)^n h(z)$$

where n is a positive integer and $h(z)$ is analytic at z_0 and $h(z_0) \neq 0$. Depending on the value of n, the local mapping by $f(z)$ at z_0 has the form.

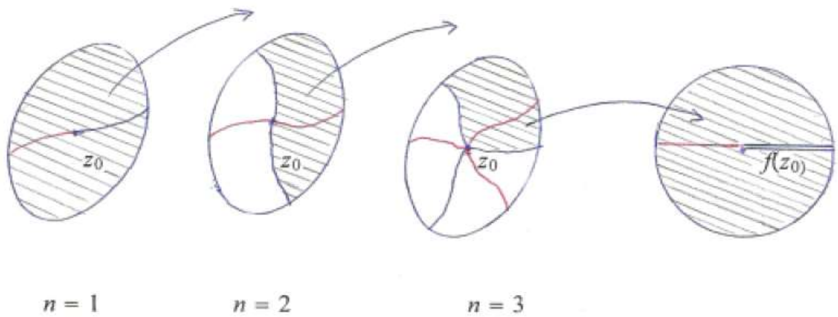

Fig. 4 The local mapping by an analytic function

Here the arcs $\gamma : s = s(x)$ are mapped by $f(z)$ onto the interval $(0, r)$, where r is the radius of the image disc, such that $f(s(x)) = x$.

They are analytic arcs (see [1],page 234) i.e. the derivative $s'(x)$ exists on $(0, r)$. Moreover, $\lim_{x \searrow 0} s'(x)$ exist.

The angles at s_0 are doubled, tripled, etc. according with $n = 2,\ n = 3,$ etc.

4**. The multiplicity of the zeros of** $\zeta_{A,\Lambda}(s)$

We proved in [3] that linear combinations of linearly independent Dirichlet functions satisfying the same Riemann type of functional equation have double zeros.

An illustration of such a zero can be seen in Fig. 5 below.

These linear combinations cannot be Euler product functions.

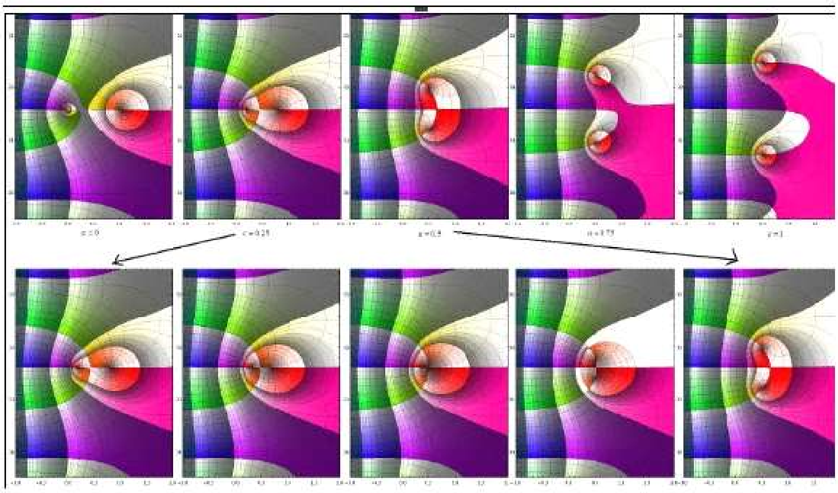

Fig. 5 A double zero of a Dirichlet function which is not Euler product

Main Theorem: ***Euler product functions do not have any multiple zero***

Proof: Suppose that $\zeta_{A,\Lambda}(s)$ is an Euler product function. It is obvious that the zeros of $\zeta_{A,\Lambda}(s)$ are located on $\Gamma_{k,j}$.

It is known (see [3]) that the zeros on $\Gamma_{k,j},\ j \neq 0$ and on $\Gamma_{0,j}$ are all simple zeros, hence we only need to deal with the zeros on $\Gamma_{k,0}$, which intertwine with $\Upsilon_{k,0}$.

Let $s = s(x)$ be the parametric equation of $\Gamma_{k,0}$ with $\zeta_{A,\Lambda}(s(x)) = x,\ s_0 = s(0)$ .Since $\zeta_{A,\Lambda}(s_0) = 0,$ the point s_0 is a pole of $\varphi(s)$. Due to (3) it is simple pole. It is obvious from

(3) that $\lim_{\sigma\to\infty}\varphi(\sigma+it) = 0 = \lim_{\sigma\to\infty}\zeta'_{A,\Lambda}(\sigma+it)$.

Suppose that s_0 is a double zero of $\zeta_{A,\Lambda}(s)$, i.e. $\zeta_{A,\Lambda}(s_0) = \zeta'_{A,\Lambda}(s_0) = 0$ and $\zeta''_{A,\Lambda}(s_0) \neq 0$. Then s_0 is located on a curve $\Gamma_{k,0}$ and $\zeta'_{A,\Lambda}(s_0) = \lim_{\sigma\to\infty}\zeta'_{A,\Lambda}(\sigma+it) = 0$, where $\sigma+it \in \Gamma_{k,0}$. Then $\zeta'_{A,\Lambda}(s)$ maps the part of $\Gamma_{k,0}$ which belongs to the pre-image of the interval $[0,1]$ onto a closed curve. Yet s_0 is an interior point of a fundamental domain of $\zeta'_{A,\Lambda}(s)$, while ∞ belongs to its boundary of that domain and such a mapping is not possible. Therefore s_0 cannot be a double zero of $\zeta_{A,\Lambda}(s)$, thus all the zeros of $\zeta_{A,\Lambda}(s)$ are simple zeros □.

It is known that Dirichlet L-series generate Euler product functions (and **the Riemann Zeta function is one of them !)** .

Thus Dirichlet L-functions (including the Riemann Zeta function) cannot have multiple zeros.

References

[1] L.V. Ahlfors, Complex Analysis, McGraw-Hill, 1979

[2] D. Ghisa, Fundamental Domains of Dirichlet Functions, Geometry, Integrability and Quantization, I.M. Mladenov, V. Pulov and A. Yoshioka Editors, Sofia, 2019, 131-160

[3] D. Ghisa, The Geometry of the Mappings by General Dirichlet Series, , APM , 2017, 7, 1-20