Weighted Approximation of Functions in L_p-norm by Baskakov-Kantorovich Operator

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Operators of Bernstein, Baskakov, Szász-Mirakjan and Meyer-König and Zeller

Classical:

$$L_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)$$

Kantorovich modification:

$$f\left(\frac{k}{n}\right) \to n \int_{D\left(\frac{k}{n}\right)} f(t)dt, \quad D\left(\frac{k}{n}\right) = \left[\frac{k}{n}, \frac{k+1}{n}\right]$$

Advantages: easier to calculate.
Durrmeyer modification:

$$f\left(\frac{k}{n}\right) \to n \int_{D} p_{n,k}(t) f(t)dt, \quad D = [0,1], [0,1), [0,\infty).$$

Advantages: good properties, easier to estimate.

The best results until 1987

Moduli of smoothness - Ditzian and Totik

$$\|\tilde{L}_n f - f\|_{p(J)} \le C\left[\omega^2_{\sqrt{\varphi}}\left(f, n^{-1/2}\right)_{p(J)} + n^{-1}\|f\|_{p(J)}\right]$$

and the big O equivalence relation (which includes a weak converse inequality)

$$\|\tilde{L}_n f - f\|_{p(J)} = O(n^{-\alpha/2}) \Leftrightarrow \omega^2_{\sqrt{\varphi}}(f, h)_{p(J)} = O(h^{\alpha}), \quad \alpha < 2,$$

where $\tilde{L}_n$ is $\tilde{B}_n$, $\tilde{V}_n$ or $\tilde{S}_n$, J is respectively $[0,1]$ or $[0,\infty)$, and $\omega^2_{\sqrt{\varphi}}(f,h)_{p(J)}$ is the second order Ditzian-Totik modulus of smoothness with varying step $\varphi(x)$, where $\varphi(x) = x(1-x)$ for the Bernstein-Kantorovich operator, $\varphi(x) = x(1+x)$ for the Baskakov-Kantorovich, $\varphi(x) = x$ for the Szász-Mirakjan-Kantorovich operator and $\varphi(x) = x(1-x)^2$ for MKZ-Kantorovich.

Moduli of smoothness - Ditzian and Totik

The second order Ditzian-Totik modulus of smoothness $\omega^2_{\sqrt{\varphi}}(f,h)_{p(J)}$ is equivalent to the next K-functional

$$K(f,t)_p = \inf\left\{\|f-g\|_p + t\left\|\varphi g''\right\|_p : g \in C^2(J)\right\}$$

i.e. there exists a constant C such that

$$C^{-1}\omega^2_{\sqrt{\varphi}}\left(f,\sqrt{t}\right)_{p(J)} \le K(f,t)_p \le C\omega^2_{\sqrt{\varphi}}\left(f,\sqrt{t}\right)_{p(J)}.$$

Bernstein operator

Classical Bernstein operator:

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right).$$

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

Kantorovicch modification of Bernstein Operator:

$$B_n^*(f,x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt.$$

Results for Bernstein-Kantorovich operator after 1987

In 1990 Berens and Xu introduced a new K-functional

$$K^*(f,t) = \inf\left\{\|f-g\|_p + t\,\|P(D)g\|_p : g \in C^2[0,1]\right\}$$

where the differential operator $P(D)$ is given by the formula

$$P(D) = \frac{d}{dx}\left(x(1-x)\frac{d}{dx}\right) = \left(\varphi(x)f'(x)\right)', \quad \varphi(x) = x(1-x)$$

and proved the direct theorem for Bernstein-Kantorovich operator using this K-functional, i.e. there exists an absolute constant C such that for every $f \in L_p[0,1]$

$$\|B_n^* f - f\| \leq CK^*\left(f, \frac{1}{n+1}\right)$$

and a week converse inequality for B_n^* in terms of $K^*(f,t)$.

Results for Bernstein-Kantorovich operator

Later, Chen and Ditzian (in 1994) proved the strong converse inequality of type B (in terms of $K^*(f,t)$), and Gonska and Zhou (the same year) of type A in terminology, suggested by Ditzian and Ivanov. So, combining the direct and converse results, we have a full equivalency between the error of approximation and the K-functional $K^*(f,t)$), i.e.

$$\|B_n^* f - f\| \sim K^*(f, \frac{1}{n}).$$

They also characterized $K^*(f,t)$) by appropriate moduli of smoothness, proving

$$K^*(f,t) \sim \omega^2_{\sqrt{\varphi}}\left(f, \sqrt{t}\right)_p + \omega\left(f,t\right)_p, \quad 1 < p \leq \infty.$$

Types of strong converse inequalities

For a sequence of uniformly bounded operators Q_n and for some sequence $\lambda(n)$ which decreases to zero, we define four types of strong converse inequalities.

A $\quad K(f,\lambda(n)) \le C\|f - Q_n f\|$ for all n (or for $n \ge n_0$).

B $\quad K(f,\lambda(n)) \le C\frac{\lambda(n)}{\lambda(k)}\{\|f - Q_n f\| + \|f - Q_k f\|\}$ for all $k \ge rn$ and some fixed $r > 1$.

C $\quad K(f,\lambda(n)) \le C\frac{1}{(r-1)n}\sum_{k=n}^{rn-1}\|f - Q_k f\|$ for all n and some $r > 1$.

D $\quad K(f,\lambda(n)) \le C\sup_{k\ge n}\|f - Q_k f\|$ for all n.

The strongest are the inequalities of type A.

Classical Baskakov Operator

The classical Baskakov operator (introduced in 1957) is defined for every function $f \in C[0, \infty)$ by

$$V_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) v_{n,k}(x)$$

where $v_{n,k}(x)$ are (so called) basic Baskakov polynomials

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

Kantorovich modifications of Baskakov Operator

In the book "Moduli of smoothness" Ditzian and Totik defined two Kantorovich modifications of V_n.
For $0 \le x < \infty$ they introduced

$$V_n^*(f,x) = \sum_{k=0}^{\infty} V_{n,k}(x)\, n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du$$

and

$$\tilde{V}_n(f,x) = \sum_{k=0}^{\infty} V_{n,k}(x)\,(n-1) \int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(u)du.$$

The reason for introducing the second one is that the first one is not a contraction ($\|V_n^*\| > 1$) and because of that it is not very suitable for approximation of functions in L_p norm for $p < \infty$.

Some definitions

$$\tilde{D} = \frac{d}{dx}\left(\psi(x)\frac{d}{dx}\right) = D\psi D, \quad D = \frac{d}{dx}.$$

$$\tilde{W}_p[0,\infty) = \{f : f, Df \in AC_{loc}(0,\infty), \tilde{D}f \in L_p[0,\infty), \lim_{x\to 0_+} xDf(x) = 0\},$$

$$L_p[0,\infty) + \tilde{W}_p[0,\infty) = \left\{f : f = f_1 + f_2, f_1 \in L_p[0,\infty), f_2 \in \tilde{W}_p[0,\infty)\right\}.$$

$$\tilde{K}(f,t)_p = \inf\left\{\|f-g\|_p + t\left\|\tilde{D}g\right\|_p : f - g \in L_p[0,\infty), g \in \tilde{W}_p[0,\infty)\right\}.$$

Direct and converse results for $\tilde{V}_n$ (2016)

For $1 < p \leq \infty$ and for every $f \in L_p[0,\infty) + \tilde{W}_p[0,\infty)$, there exist absolute constants $R, C > 0$ such that for every natural $l \geq Rn$

$$C^{-1}\|\tilde{V}_n f - f\|_p \leq \tilde{K}\left(f, \frac{1}{n}\right)_p \leq C\frac{l}{n}\left(\|\tilde{V}_n f - f\|_p + \|\tilde{V}_l f - f\|_p\right).$$

The left inequality is true for $p = 1$ as well.
Another way to state this is: there exists an integer k such that

$$\tilde{K}\left(f, \frac{1}{n}\right)_p \sim \|\tilde{V}_n f - f\|_p + \|\tilde{V}_{kn} f - f\|_p, \quad p > 1.$$

Some open questions

1. Proof of the converse inequality of type B for $p = 1$.

2.Proof of the converse inequality of type A.

3.Characterization of the K-functional by appropriate moduli of smoothness.
(2017) For $1 < p < \infty$ and for every

$$f(x) \in L_p[0,\infty) + \tilde{W}_p[0,\infty)$$

such that

$$\lim_{x\to\infty} \psi(x) f'(x) = 0$$

we have

$$\tilde{K}\,(f,t)_p \sim \omega^2_{\sqrt{\psi}}(f,\sqrt{t})_p + tE_0(f)_p$$

where

$$E_0(f)_p = \inf_c \|f - c\|.$$

The best results about weighted approximation by Kantorovich operators

Moduli of smoothness - Ditzian, Totik

Theorem

Let $w^*(x) = x^{\gamma(0)}(1+x)^{\gamma(\infty)}$ where $\gamma(\infty)$ is arbitrary and $-1/p < \gamma(0) < 1 - 1/p$ for $1 \le p \le \infty$, $w^* f \in L_p[0,\infty)$ and either $1 \le p \le \infty$ and $\alpha < 1$, or $1 < p < \infty$ and $\alpha \le 1$. Then for $\tilde{V}_n^*$ the next equivalency is true.

$$\left\| w^* \left(\tilde{V}_n^* f - f \right) \right\|_p = O\left(n^{-\alpha} \right) \Leftrightarrow \left\| w^* \Delta^2_{h\sqrt{\psi}} f \right\|_{L_p[2h^2,\infty)} = O\left(h^{2\alpha} \right).$$

Here $\| \circ \|_p$ and $\| \circ \|_{L_p(J)}$ stand for the usual L_p-norm respectively on $[0,\infty)$ and the interval J, $\psi(x) = x(1+x)$ and

$$\Delta^2_{h\sqrt{\varphi(x)}}(f,x) = f\left(x - h\sqrt{\varphi(x)}\right) - 2f(x) + f\left(x + h\sqrt{\varphi(x)}\right).$$

Baskakov - Kantorovich operator $\tilde{V}_n$ - definitions

$$w(x) = (1+x)^{\alpha}, \qquad \alpha \in \mathbb{R}.$$

$$\tilde{D} = \frac{d}{dx}\left(\psi(x)\frac{d}{dx}\right)$$

$$L_p(w) = \{f : wf \in L_p[0,\infty)\}$$

$$W_p(w) = \left\{f : w\tilde{D}f \in L_p[0,\infty),\ \lim_{x\to 0^+} \psi(x)f'(x) = 0\right\}, \quad \alpha < 0$$

$$W_p(w) = \left\{f : w\tilde{D}f \in L_p[0,\infty),\ \lim_{x\to 0^+,\infty} \psi(x)f'(x) = 0\right\}, \quad \alpha > 0$$

$$L_p(w) + W_p(w) = \left\{f : f = f_1 + f_2,\ f_1 \in L_p(w),\ f_2 \in W_p(w)\right\}.$$

$$K_w(f,t)_p = \inf\left\{\|w(f-g)\|_p + t\left\|w\tilde{D}g\right\|_p : f-g \in L_p(w),\ g \in W_p(w)\right\}$$

Baskakov - Kantorovich operator $\tilde{V}_n$ (weighted appr.)

Theorem

For $1 \leq p \leq \infty$ there exists a positive constant C such that for every $n > |\alpha|$, $n \in \mathbb{N}$, and for all functions $f \in L_p(w) + W_p(w)$ there holds

$$\left\| w\left(\tilde{V}_n f - f\right)\right\|_p \leq CK_w\left(f, \frac{1}{n}\right)_p.$$

For $1 < p \leq \infty$ there exist absolute constants $R, C > 0$ such that for every natural $l \geq Rn$ and for all functions $f \in L_p(w) + W_p(w)$ there holds

$$K_w\left(f, \frac{1}{n}\right)_p \leq C\frac{l}{n}\left(\left\| w\left(\tilde{V}_n f - f\right)\right\|_p + \left\| w\left(\tilde{V}_l f - f\right)\right\|_p\right).$$

Another way to state the Theorem is: there exists an integer k such that

$$K_w\left(f, \frac{1}{n}\right)_p \sim \left\| w(\tilde{V}_n f - f)\right\|_p + \left\| w(\tilde{V}_{kn} f - f)\right\|_p, \quad p > 1.$$

Thank you for your attention!