The Banach Gelfand Triple and Fourier Standard Spaces

Hans G. Feichtinger, Univ. Vienna
hans.feichtinger@univie.ac.at
www.nuhag.eu

Talk held at AnMath Conference
Budapest, August 16th, 2019

official abstract I

Central objects of *classical Fourier Analysis* are the Fourier transform (often just viewed as an integral transform defined on the Lebesgue space $\boldsymbol{L}^1(\mathbb{R}^d)$), convolution operators, periodic and non-periodic functions in L^p-spaces and so on. *Distribution theory* widens the scope by allowing larger families of Banach spaces of functions or generalized functions and extending many of the concepts to this more general setting. Although, according to A. Weil the natural setting for Fourier Analysis (leading to the spirit of Abstract Harmonic Analysis: AHA) most of the time one works in the setting of the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ of rapidly decreasing functions and its dual space, the *tempered distributions*. In this setting weighted $\boldsymbol{L}^2$-spaces and Sobolev spaces correspond to each other in a very natural way.

official abstract II

In this talk we will summarize the advantages with respect to the level of technical sophistication and theoretical background which is possible when one uses instead of the Schwartz-Bruhat space $\mathcal{S}(G)$ the Segal algebra $S_0(G)$ and the resulting Banach Gelfand Triple (S_0, L^2, S_0'), which appears to be suitable for the description of most problems in AHA as well as for many engineering applications (this part is beyond the scope of the current talk). Among others the use of *Wiener amalgam spaces* $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)$ and *modulation spaces* $M^{p,q}$ (introduced by the author in the 1980s) belong to a comprehensive family of Banach spaces $(B, \|\cdot\|_B)$ embedded between $\boldsymbol{S}_0$ and $\boldsymbol{S}_0'$, which we call *Fourier Standard Spaces*. These spaces have a *double module structure*, with respect to convolution by $\boldsymbol{L}^1$-functions and pointwise multiplication with functions from the Fourier algebra $\mathcal{F}\boldsymbol{L}^1$.

official abstract III

The most interesting examples are Banach spaces of (generalized) functions containing $S_0(G)$ (or just $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ for the Euclidean case) as a dense subspace and such that time-frequency shifts $f \mapsto \pi(t,\omega)f$ are isometric on $(B, \|\cdot\|_B)$, where

$$\pi(t,\omega)f(x) = e^{2\pi i\omega\cdot x}f(x-t), \quad x,t,\omega \in R^d,$$

or the dual of such a space.

There is a long list of examples of such spaces. Any *reflexive* Banach space (of tempered) distributions belongs to this family, hence in some sense this family allows to derive statements typically valid for $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 < p < \infty$.

Personal Background

- Educated in Abstract Harmonic Analysis (H. Reiter);
- I got interested in Function Spaces (H. Triebel);
- Started to do Numerical Harmonic Analysis in College Park/MD (since 1989, using MATLAB);
- Realizing a number of real world applied projects, with colleagues from image processing, communication theory, astronomy, geophysics, etc.;
- Current goal: CONCEPTUAL HARMONIC ANALYSIS (sythesis of ABSTRACT and NUMERICAL resp. COMPUTATIONAL harmonic analysis);
- Using MATLAB to do explorative simulations, but also to develop efficient algorithms.

Motivation concerning natural function spaces I

In this talk I would like to take a fresh look at the various function spaces which arose from Fourier Analysis. It is true that they have shaped the early phase of Functional Analysis (with strong contributions from Hungary). Speaking in Budapest I assume that the audience is familiar with the classical theory of Fourier Series and Fourier transforms.
Let me start to mention a couple of function spaces which play a role in Fourier analysis. The most natural function space on the torus is certainly $\left(\boldsymbol{C}(\mathbb{T}), \|\cdot\|_\infty\right)$, the space of continuous, complex-valued functions on $\mathbb{T}$ endowed with the sup-norm.

$$\|f\|_\infty := \max_{x\in\mathbb{T}} |f(x)|,$$

corresponding to uniform convergence.

Motivation concerning natural function spaces II

It is used to formulate the Theorem of (Stone)-Weierstrass: *Polynomial functions form a DENSE subalgebra* of $\left(\boldsymbol{C}(\mathbb{T}), \|\cdot\|_\infty\right)$.
In fact also the Riemanian integral is well defined on this space, defined as the strong limit of linear combinations of discrete measures (the finite Riemannian sums), with the speed of convergence depending on (the smoothness of) f.
With the work of H. Lebesgue the notation of an integral has been pushed to its natural limit, i.e. to what is nowadays called the *Lebesgue space* $\left(\boldsymbol{L}^1(\mathbb{T}), \|\cdot\|_1\right)$ (of equivalence classes of measurable functions). By the Riesz representation theorem it is a closed subspace of the dual of $\left(\boldsymbol{C}(\mathbb{T}), \|\cdot\|_\infty\right)$.
Looking from the Hilbert space perspective we observe (following F. Riesz) that the Fourier transform, defined properly on $\boldsymbol{L}^2(\mathbb{T})$, defines a unitary isomorphism onto $(\ell^2(\mathbb{Z}), \|\cdot\|_2)$, and a unitary automorphism for $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$.

Motivation concerning natural function spaces III

From there it was natural to look for other exponents, which lead to the theory of $\boldsymbol{L}^p$-spaces. They are all Banach spaces, with the dual being just another member of this family, with the well-known relation $1/q + 1/p = 1$.
Next one can look for other case, where $\left(\boldsymbol{L}^p(\mathbb{T}), \|\cdot\|_p\right)$ is mapped onto $\ell^r(\mathbb{Z})$, but there is no other such pair. The best we can say is the Hausdorff-Young inequality with allows to estimate the range of the Fourier transform applied to $\left(\boldsymbol{L}^p(\mathbb{T}), \|\cdot\|_p\right)$ (with $1 \leq p \leq 2$) by the dual space $\ell^q(\mathbb{Z})$.
Consequently it becomes interesting to ask for the space of all pointwise multipliers (on the FT side) from $\mathcal{F}(\boldsymbol{L}^p(\mathbb{T}))$ into $\mathcal{F}(\boldsymbol{L}^r(\mathbb{T}))$, for different values of $p, r \in [1, \infty]$, i.e. turn to the question of *Fourier multipliers*.

Motivation concerning natural function spaces IV

An important family of such multipliers is concerned with multipliers from $\mathcal{F}(\boldsymbol{L}^1(\mathbb{T}))$ into $\ell^1(\mathbb{Z})$, the so-called *summability kernels*.

These Fourier multipliers are in fact much more important, because they require functions with good decay, so that they are integrable (hence the Fourier inversion formula makes sense), but in order to ensure that one has convergence on the "time-side" one also needs that these corresponding family (inverse Fourier transforms of the summability kernels) form a "*Dirac sequence*", hence in particular are uniformly bounded in $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.

Lorentz and Orlicz Spaces

One may of course ask whether it is possible to have a refinement of the *scale of* $\boldsymbol{L}^p$-spaces, i.e. generate spaces with more parameters. In retrospect one can say that the so-called *Lorentz spaces* are those spaces which are obtained by applying general *real interpolation methods* with parameters (θ, q) (with $1 \leq q < \infty$) to the ordinary $\boldsymbol{L}^p$-spaces. Usually the resulting spaces are denoted by $\boldsymbol{L}(p, q)$.

Another family, the so-called *Orlicz spaces* arises by replacing the exponential function $u \mapsto |u|^p$ in the definition of $\boldsymbol{L}^p$-spaces by another function. To define a suitable norm on these spaces turns out to be non-trivial.

Nevertheless all the resulting spaces have the property of being *rearrangement invariant* (with respect to measure preserving transformations).

Abstract Harmonic Analysis

Abstract Harmonic Analysis provides a good framework for Fourier Analysis - even for applied Fourier Analysis - when it comes to a comparison if different, still quite similar in an ABSTRACT sense.

There is always a commutative group, and corresponding translation operators of the functions on that group, and a *dual group*, which consists of the (bounded and continuous) joint eigenvectors for these (unitary) operators on $\left(\boldsymbol{L}^2(G), \|\cdot\|_2\right)$.

Engineers make a distinction between discrete and continuous variables, between periodic and non-periodic signals, while AHA talks about compact or discrete groups resp. dual groups. The discrete + periodic case is naturally identified with the "finite" case, i.e. with signals over $\mathbf{Z}_N$ (group of unit roots of order N).

Discrete and Fast Fourier Transform

For this finite case the situation is quite simple:
The Discrete Fourier Transform (DFT) can be viewed as a change of basis, with the natural basis of unit vectors (obtained by applying the cyclic shift operator to any one of them) is replaced by the "pure frequencies", resp. the DFT matrix.

This matrix is (up to the scaling factor $\sqrt{N}$) a unitary matrix. It can be also viewed as a *Vandermonde Matrix*, describing the mapping from a vector $[a_1, ..., a_N]$ to the value of the polynomial $p_a(z) = \sum_{k=1}^{N} a_k z^{k-1}$ at the unit roots of order N, taken in the mathematical positive sense, starting with $1 = \omega_N^0$.

From there many properties (like the sampling theorem) are easily derived! Also Poisson's formula is valid in this setting.

Non-compactness: Problems for the Euclidean Situation

Let us shortly reflect that situation concerning

Fourier Analysis over the Euclidean setting.

DIRECT FOURIER TRANSFORM: $\widehat{f}(s) = \int_{\mathbb{R}^d} f(t)\overline{\chi_s(t)}dt$
INVERSE FOURIER TRANSFORM: $f(t) = \int_{\mathbb{R}^d} \widehat{f}(s)\chi_s(t)ds$
Compared to the finite dimensional setting we have new problems:

1. The pure frequencies or *characters* $\chi_s(t) := exp(2\pi is \cdot t)$ do not belong to the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, leave alone to $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.
   So do they HAVE (?) a Fourier transform?
2. Even for $f \in \boldsymbol{L}^1(\mathbb{R})$ it is not guaranteed that the Fourier transform $\widehat{f}$ belongs to $\boldsymbol{L}^1(\mathbb{R})$ as well. This is why in the classical setting *summability kernels* have to be invoked.

Ways to deal with this situation I

Pure mathematicians have spent a lot of energy in trying to develop methods of analysis that allow to turn the intuition (up to the point that it gets almost lost, e.g. in the discussion of *spectral analysis*!) into correct mathematical statements, leading to highlights such as the results of L. Carleson or the theory of tempered distributions by L. Schwartz.
Of course it is not enough to NAME the inversion formula as such. One has to show in fact for which domains and target spaces the Fourier transform is well defined and the "inverse transform" is taking correctly the role of an "inverse mapping".

Looking ahead: $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ (and its dual) as an appropriate setting for such a claim, allowing to work with ordinary *Riemann integrals*, but still in a distributional setting.

Ways to deal with this situation II

Engineers and physicists rely more their "intuition" and avoid the technical approach (using Lebesgue's theory, topological vector spaces, heavy functional analytic arguments) and argue on a more symbolic level.
In this spirit one finds claims such as

- The family $(\delta_x)_{x\in\mathbb{R}}$ is a continuous ONB for $\boldsymbol{L}^2(\mathbb{R})$;
- The family $(\chi_s)_{s\in\mathbb{R}}$ is a continuous ONB for $\boldsymbol{L}^2(\mathbb{R})$;
- The (forward and inverse) Fourier transform describe a change of bases between these two natural bases.

One of the formulas appearing frequently in this context is:

$$\int_{-\infty}^{\infty} e^{2\pi i s x}\, ds = \delta(x).$$

Ways to deal with this situation III

Transition from periodic to non-periodic

Another element of concern in this context is the usual description and motivation for formulas describing the "continuous" (forward and inverse) transform.
Starting from the periodic case, which can be well described using the rather concrete complete orthonormal system of pure frequencies from a lattice (compatible with the periodicity of the functions to be expanded, so with a spacing which is the inverse of the periodicity) the argument is to let the period go to infinity and to use some vague arguments "justifying" the continuous formulas.

Again, $\boldsymbol{S}_0'$ (with the w^*-topology) can be used to provide a mathematical correct justification, but this goes beyond the scope of this talk.

The Schwartz Setting

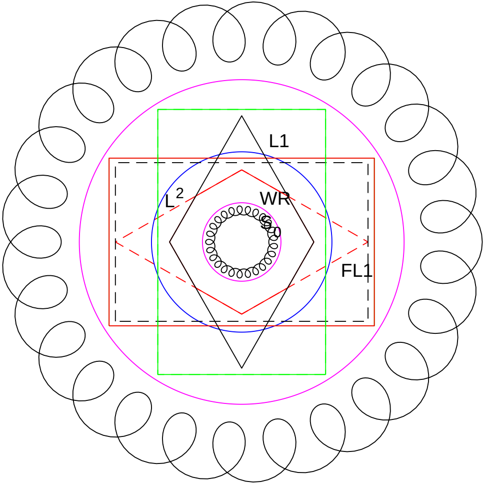


Figure: The classical setting

BUPUs: Bounded Uniform Partitions of Unity

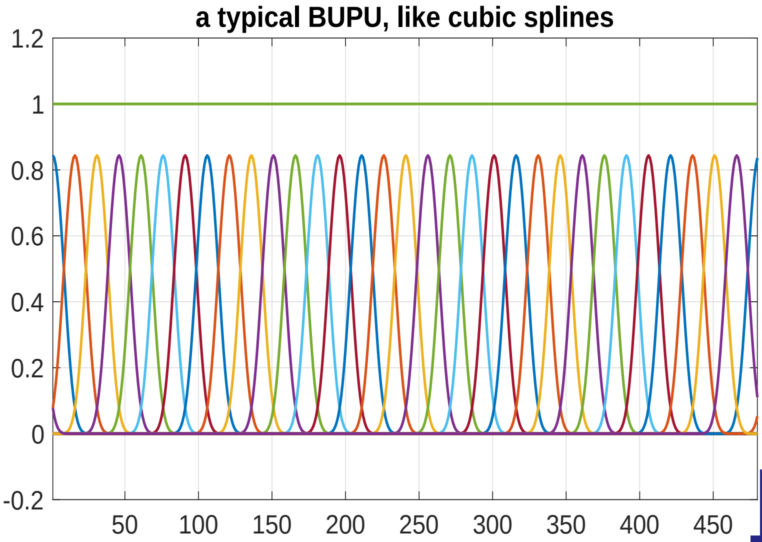
Definition

A bounded family $\Psi = (\psi_i)_{i \in I}$ in a Banach algebra $(\mathbf{A}, \|\|_A)$ is called a Bounded Uniform Partition of Unity in $(\mathbf{A}, \|\|_A)$ (a BUPU in $\mathbf{A}$, for short), if there are a relatively separated[a] family $X = (x_i)_{i \in I}$ and some $R > 0$ such that

1. $\mathrm{supp}(\psi_i) \subseteq B_R(x_i)$ for each $i \in I$, and
2. $\sum_{i \in I} \psi_i(x) = 1$ for all $x \in \mathbb{R}^d$

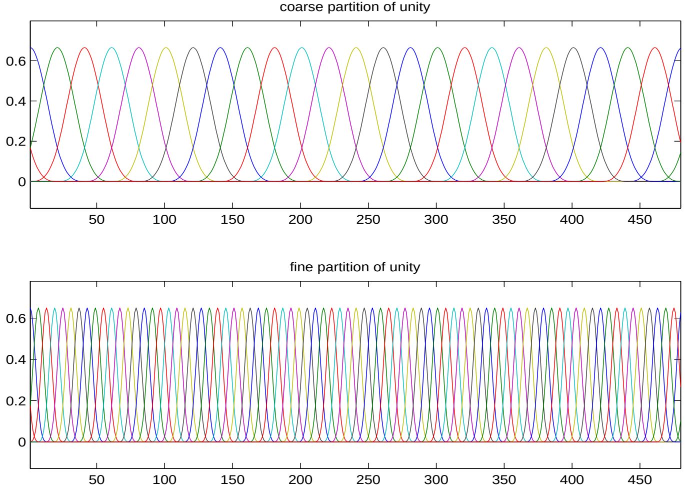
[a] i.e. a finite union of γ-separated sets.

The most useful variant are the so-called regular BUPUs with $I = \Lambda \lhd \mathbb{R}^d$, a lattice, with $\psi_i = T_\lambda(\varphi)$, for some $\varphi = \varphi_0$, such as a cubic B-spline.

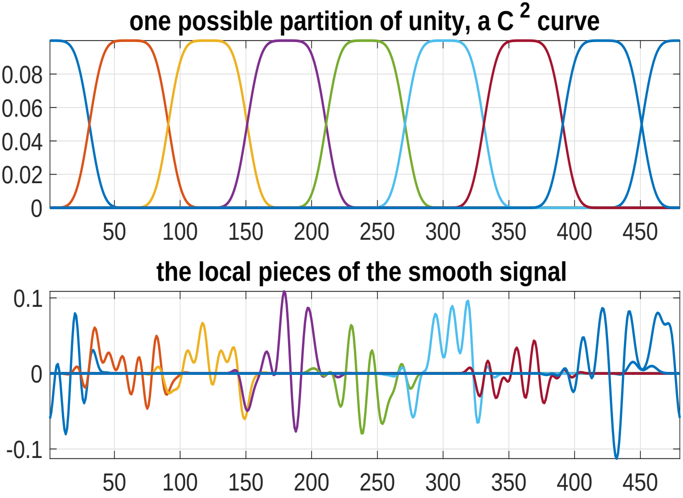


a typical BUPU, like cubic splines
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
50
100
150
200
250
300
350
400
450

BUPUs: Bounded Uniform Partitions of Unity



Wiener Amalgam Norms Vizualized



Recalling the Wiener Amalgam Concept

We recall the concept of BUPUs ideally as translates along a lattice ($T_\lambda \varphi$), with compact support and a certain amount of smoothness, perhaps cubic B-splines.
The Wiener amalgam space $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$ is defined as the set

$$\left\{ f \in \boldsymbol{B}_{loc} \,|\, \|f | \boldsymbol{W}(\boldsymbol{B}, \ell^q)\| := \left(\sum_{\lambda \in \Lambda} \|f \cdot T_\lambda \varphi\|_{\boldsymbol{B}}^q \right)^{1/q} \right\}.$$

There are many "natural results" concerning Wiener amalgam spaces, namely coordinatewise action, e.g.

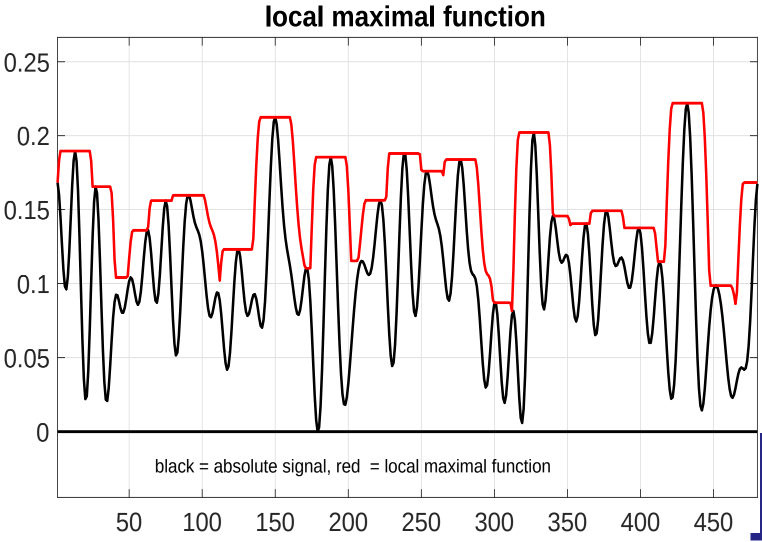
- duality (if test functions are dense and $q < \infty$);
- convolution and pointwise multiplication;
- interpolation (real or complex).

The classical Wiener algebra

In the context of Segal algebras the space $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ (the closure of the test functions in $\boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)(\mathbb{R}^d)$) appears early on in the literature (e.g. in the work of N. Wiener on the Tauberian Theorems), also known as Wiener's algebra.
We can describe it as the space of continuous, complex-valued functions whose absolute value has a finite upper Riemannian sum (over the full Euclidean space $\mathbb{R}^d$).
It can be shown to be the smallest Segal algebra which is in addition also allows pointwise multiplications with bounded, continuous functions or just by elements $h \in \boldsymbol{C}_0(\mathbb{R}^d)$. The classical norm in Reiter's book is

$$\|f\|_{\boldsymbol{W}}(\mathbb{R}) = max_{z\in\mathbb{R}} \sum_{k\in\mathbb{Z}} max_{y\in[k,k+1]}|f(y-z)| < \infty.$$

The local maximal function



The Classical Amalgams $W(L^p, \ell^q)$

The first observation (see the survey article of Fournier/Stewart in Bull. AMS from 1985) is that one has nice inclusion relations for amalgam spaces, but in different directions, depending on the local (p) resp. global argument (q).
There are also Hausdorff-Young Theorems for Wiener Amalgams:

Theorem

1. $\mathcal{F}(W(L^p, \ell^q)) \subseteq W(L^{q'}, \ell^{p'})$, *for* $1 \leq p, q \leq 2$;
2. $\mathcal{F}(W(\mathcal{F}L^p, \ell^q)) \subseteq W(\mathcal{F}L^q, \ell^p)$, *for* $1 \leq q \leq p \leq \infty$;
3. *In particular, the spaces* $W(\mathcal{F}L^p, \ell^p)(\mathbb{R}^d)$ *form an increasing family of Fourier invariant Banach spaces of (mild) distributions, equal to* $L^2(\mathbb{R}^d)$ *for* $p = 2$, *and with*

$$W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d) = S_0(\mathbb{R}^d) \subset W(\mathcal{F}L^p, \ell^p)(\mathbb{R}^d) \subset S_0'(\mathbb{R}^d).$$

Homogeneous Banach Spaces

In his book on Fourier Analysis (which first appeared in 1968) Y. Katznelson describes homogeneous Banach spaces as one possible generalization of ordinary $\boldsymbol{L}^p(G)$-spaces.

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ of locally integrable functions is called a *homogeneous Banach space* on $\mathbb{R}^d$ if it satisfies

1. $\|T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}} \quad \forall f \in B, x \in \mathbb{R}^d$;
2. $\|T_x f - f\|_{\boldsymbol{B}} \to 0$ for $x \to 0, \quad \forall f \in \boldsymbol{B}$.

EX: $\boldsymbol{B} = \boldsymbol{L}^p(\mathbb{R}^d)$ for $1 \le p < \infty$, or *reflexive BF spaces*.
Via vector-valued integration one derives

$$\|g * f\|_{\boldsymbol{B}} \le \|g\|_{\boldsymbol{L}^1} \|f\|_{\boldsymbol{B}}, \quad \forall g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B}. \tag{1}$$

Segal Algebras

For his work related to *spectral analysis* over LCA groups H. Reiter introduced in his book the so-called Segal algebras.
Using the terminology of Homogeneous Banach Spaces one can abbreviate their characterization as follows:

Definition

A homogeneous Banach space is called a Segal Algebra (in the sense of H. Reiter) if it has the additional properties

1. $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is continuously embedded into $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$;
2. $\boldsymbol{B}$ is dense in $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$.

Alternatively one can characterize them as dense, essential Banach ideals in the Banach algebra $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$.

Banach Module Terminology

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module* over a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\boldsymbol{A} \times \boldsymbol{B}$ into $\boldsymbol{B}$ bilinear and associative, such that

$$\|a \bullet b\|_{\boldsymbol{B}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{B}} \quad \forall\, a \in \boldsymbol{A}, b \in \boldsymbol{B}, \tag{2}$$

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}. \tag{3}$$

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach ideal* in (or within, or of) a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is continuously embedded into $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$, and if in addition (2) is valid with respect to the internal multiplication inherited from $\boldsymbol{A}$.

Double Modules

Over the years it has turned out that for the discussion of translation invariant operators pointwise multiplicative structures are helpful, and conversely. Hence we will take a look at Banach spaces having a **double module structures**.
Usually we refer to convolution operators as smoothing operators or mollifiers. In contrast pointwise multipliers (such as multiplication with a summability kernel) improves locality, so they are sometimes also called "localizers".
If the convolution structure comes from $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, with the usual convolution (extending naturally to the convolution of bounded, regular Borel measures), we do not have an identity element, but we have *bounded approximate units* (Dirac sequences), via dilation. Their Fourier transforms (stretched summability kernels) provide BAUs for $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$.

Minimality of $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$

Within this family of double modules $\left(\left(L^1(\mathbb{R}^d), \|\cdot\|_1\right)\right.$ acts via convolution, $\left(\mathcal{F}L^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}L^1}\right)$ via pointwise multiplication) the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest member.

Theorem

There is a smallest member in the family of all TF-homogeneous Banach spaces, namely the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right) = W(\mathcal{F}L^1, \ell^1)(\mathbb{R}^d)$.
It is also a modulation space, i.e. it can be characterized as

$$S_0(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \mid V_g f \in L^1(\mathbb{R}^{2d})\}$$

for some/any non-zero Schwartz function g*, with norm*

$$\|f\|_{S_0} = \|V_g f\|_1.$$

Justifying the properties of the family

There is a large number of results concerning $\boldsymbol{S}_0(\mathbb{R}^d)$ (defined for any dimension, but in fact for any LCA group):

$$\mathcal{F}_{\mathcal{G}} \boldsymbol{S}_0(G) = \boldsymbol{S}_0(\widehat{\mathcal{G}}).$$

There is a tensor product property, namely

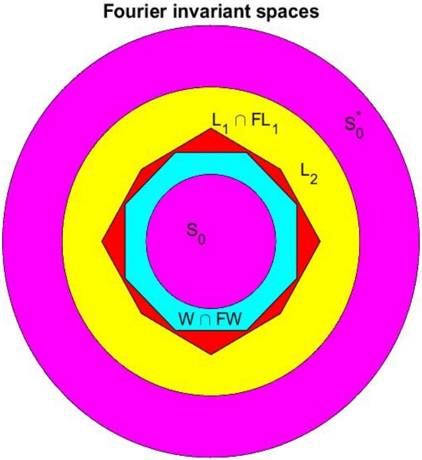
$$\boldsymbol{S}_0(\mathbb{R}^{2d}) = \boldsymbol{S}_0(\mathbb{R}^d) \widehat{\otimes} \boldsymbol{S}_0(\mathbb{R}^d).$$

Multipliers are easily characterized:

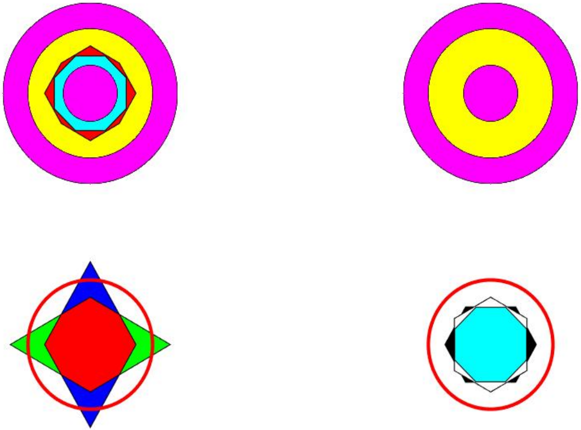
Theorem

The continuous linear operators from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$ are exactly the convolution operators by "kernels" $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$, with equivalence of norms (the operator norm of the convolution operator, resp. translation invariant linear system) and the $\boldsymbol{S}_0'$-norm of the corresponding convolution kernel $\|\sigma\|_{\boldsymbol{S}_0'}$.

Fourier Invariant Spaces



Fourier Invariant Spaces II



Minimal Spaces

Given any Fourier standard space one can apply product-convolution and convolution-product operators, which both approximate (in some sense) the identity operators: Convolution with elements of a Dirac sequence (bounded in $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$) acts as *mollifier*, while multiplication with (essentially) summability kernels (belonging to $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$, bounded there) provide *localizers*. Combining them we obtain operators of the form

$$f \mapsto h \cdot (g * f) \quad \text{resp.} \quad f \mapsto g * (h \cdot f),$$

with $g, h \in \boldsymbol{S}_0(\mathbb{R}^d)$.
Since these *regularizing* operators are (individually) bounded operators from $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ to $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ they also map $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ into $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

SO-BGTr

The rest of this talk will be concerned with the so-called Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. It can be defined over general LCA groups.
Together with its dual the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ provides an appropriate setting for (unweighted) Banach spaces of "functions" suitable for Fourier Analysis and applications (in the spirit of H. Triebel).

For comparison the Banach Gelfand Triple situation can be compared with the inclusion of number systems $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

We can only outline some of the general topics in the subsequent slides.

Fourier Standard Spaces

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ of (tempered) distributions is called a **Fourier standard space** if it satisfies the following conditions:

1. $\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d)$,
2. $\boldsymbol{B}$ is translation and modulation isometrically invariant, i.e.
$$\|M_\omega T_x f\|_{\boldsymbol{B}} = \|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \lambda = (x, \omega).$$
3. The Fourier algebra $\boldsymbol{A}$ defines pointwise multipliers on $\boldsymbol{B}$:
$$\|h \cdot f\|_{\boldsymbol{B}} \leq \|h\|_{\boldsymbol{A}} \|f\|_{\boldsymbol{B}}, \quad h \in \boldsymbol{A} := \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d),\ f \in \boldsymbol{B}.$$
4. $\boldsymbol{B}$ is a Banach convolution module over $\boldsymbol{L}^1(\mathbb{R}^d)$, with
$$\|g * f\|_{\boldsymbol{B}} \leq \|g\|_1 \|f\|_{\boldsymbol{B}}, \quad g \in \boldsymbol{L}^1(\mathbb{R}^d),\ f \in \boldsymbol{B}.$$

Some Remarks

Remark

Assuming (3) and (4) one can start equivalently from the situation

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

Remark

If $\mathcal{S}(\mathbb{R}^d)$ (hence $\boldsymbol{S}_0(\mathbb{R}^d) \supset \mathcal{S}(\mathbb{R}^d)$) is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ then (2) implies automatically (3). In fact, from the continuity of translation (and modulation) one could obtain the "integrated group action" by standard vector-valued integration methods.

Remark

Combining the two actions into the group action of the *reduced Heisenberg group* (via the so-called Schrödinger representation) one has in fact *Heisenberg modules*.

Operations within the family I

The setting of Fourier standard spaces allows to treat a large number of "derived spaces" in a unified viewpoint. In fact, most of the operations which play a role in Fourier Analysis, but also in Gabor or Time-Frequency Analysis can be applied to Fourier Standard Spaces.
Hence one can treat those questions in a more systematic way, avoiding the purely technical questions of integrability and concentrate on various interesting, and sometimes completely overlooked questions.

Constructions within the FSS Family

1. Taking **Fourier transforms**;
2. Conditional dual spaces, i.e. the **dual space** of the closure of $\mathbf{S}_0(G)$ within $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$, i.e. only for minimal spaces;
3. With two spaces $\mathbf{B}^1, \mathbf{B}^2$: take **intersection or sum**
4. forming **amalgam spaces** $\mathbf{W}(\mathbf{B}, \ell^q)$; e.g. $\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)$;
5. forming **modulation spaces** $\mathbf{M}^{p,q} = \mathcal{F}(\mathbf{W}(\mathcal{F}\mathbf{L}^p, \ell^q))$;
6. defining pointwise or convolution **multipliers**;
7. using complex (or real) **interpolation methods**, so that we get the spaces $\mathbf{M}^{p,p} = \mathbf{W}(\mathcal{F}\mathbf{L}^p, \ell^p)$ (all Fourier invariant);
8. Applying **automorphism** such as dilations, rotations;
9. any **metaplectic** image of such a space, e.g. the **fractional Fourier transform**.

Compactness in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$

Lemma

$\boldsymbol{S}_0(\mathbb{R}^d)$ is a dense subspace of a FouSS $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ if and only if the corresponding operators converge to f for any $f \in \boldsymbol{B}$.

We call such space minimal double modules.
In a paper published in Analysis Mathematica in 1982 I have shown a result which implies in our context:

Theorem

A bounded and closed subset M of a FouSS $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is compact if and only if it is equicontinuous and tight, i.e. for any $\varepsilon > 0$ there exists $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ and $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ such that

$$\|g * f - f\|_{\boldsymbol{B}} \leq \varepsilon \quad \forall f \in M;$$

$$\|h \cdot f - f\|_{\boldsymbol{B}} \leq \varepsilon \quad \forall f \in M.$$

Fourier Multipliers I

Let $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ and $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$ be two Fourier standard spaces (think of $\boldsymbol{B}^1 = \boldsymbol{L}^p$ and $\boldsymbol{B}^2 = \boldsymbol{L}^q$). Then we can define the space of **multipliers** from $\boldsymbol{B}^1$ to $\boldsymbol{B}^2$.

Definition

$$\boldsymbol{M}_{\boldsymbol{B}^1,\boldsymbol{B}^2} := \{T : \boldsymbol{B}^1 \to \boldsymbol{B}^2,\ T \circ T_x = T_x \circ T \text{ for all } x \in \mathbb{R}^d\}.$$

Given the properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ and its dual $\boldsymbol{S}_0'(\mathbb{R}^d)$ one can verify that this is another Fourier standard space. The generalized Fourier transform in the sense of $\boldsymbol{S}_0'(\mathbb{R}^d)$ maps $\boldsymbol{M}_{\boldsymbol{B}^1,\boldsymbol{B}^2}$ onto the space of pointwise multipliers between the Fourier images $\mathcal{F}\boldsymbol{B}^1$ and $\mathcal{F}\boldsymbol{B}^2$, mapping "*convolution kernels*" into "*transfer functions*" (engineering terminology).
For $\boldsymbol{B}^1 = \boldsymbol{L}^p(\mathbb{R}^d) = \boldsymbol{B}^2$ this is exactly the classical space of Fourier multipliers.

Fourier Multipliers II

Since $\boldsymbol{S}_0(\mathbb{R}^d)$ is a subspace of the so-called *quasi-measures* (according to M. Cowling they can be identified with the dual space for the space of test functions from the Fourier algebra with compact support) this last result implies that the action of a multipliers T (at least on test functions) can be described as the convolution with some element from $\boldsymbol{S}_0'(\mathbb{R}^d)$: $Tf(x) = \sigma(T_x f^{\checkmark})$. The reader can find a lot of other examples about multiplier spaces between $\boldsymbol{L}^p$-spaces in the book of R. Larsen (1972).

Questions arising from local/global considerations?

There are at least two major type of questions which one can ask, related to the possibility of creating new spaces within the family. The key constructions have to do with

Wiener amalgam spaces of the form $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$;

Let us recall that for the construction of Wiener amalgam spaces we only need the possibility of applying a BUPU (a bounded partition of unity), in our case boundedness refers to $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$.
Moreover there is a clear chain of proper inclusions with the scale of space of the form $\boldsymbol{W}(\boldsymbol{B}, \ell^p)$, with increasing p, with $\boldsymbol{W}(\boldsymbol{B}, \ell^1)$ as the smallest, certainly embedded into $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, and contained in $\boldsymbol{W}(\boldsymbol{B}, \ell^\infty)$.

Lower and Upper Index of a Function Space

We define a *lower* resp. *upper index* for a Fourier Standard space:

Definition

$$\mathrm{low}(\boldsymbol{B}) := \sup\{r \,|\, \boldsymbol{B} \subseteq W(\boldsymbol{B}, \ell^r)\}.$$

Definition

The *upper index* of $\boldsymbol{B}$ is defined as follows:

$$\mathrm{upp}(\boldsymbol{B}) := \inf\{s \,|\, W(\boldsymbol{B}, \ell^s) \subseteq \boldsymbol{B}\}.$$

For $\boldsymbol{B} = \mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d)$ or $\boldsymbol{B} = \mathcal{F}\boldsymbol{L}^q(\mathbb{R}^d)$ $(1/p + 1/q = 1)$, with $1 \leq p \leq 2$ one has $\mathrm{low}(\boldsymbol{B}) = p$ and $\mathrm{upp}(\boldsymbol{B}) = q$.

Intersections, tempered L^p-spaces

Clearly the intersection (but also the sum, related by duality) of two FouSS is again FouSS, with the norm for $\boldsymbol{B}^1 \cap \boldsymbol{B}^2$:

$$\|f\|_{\boldsymbol{B}^1 \cap \boldsymbol{B}^2} := \|f\|_{\boldsymbol{B}^1} + \|f\|_{\boldsymbol{B}^2}.$$

Since it still has to contain $\boldsymbol{S}_0(\mathbb{R}^d)$ it cannot be trivial. A classical, non-trivial example is found in the work of K. McKennon in the early 70th, who - from the point of view of FouSS - was studying the Banach algebra $\boldsymbol{L}_p^t := \boldsymbol{L}^p \cap \mathbf{Conv}_p$ of all elements of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) = \left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$ which at the same time define bounded "multipliers" on $\left(\boldsymbol{L}^p(G), \|\cdot\|_p\right)$.
The interesting finding of his work was (valid at least for Abelian groups): The space of multipliers of this new space can be identified with the $\boldsymbol{L}^p$-multipliers.

The Kernel Theorem

So-called Kernel Theorems can be viewed as the continuous analogue of a matrix representation of a general linear operator. In the finite dimensional setting the situation is quite simple. Every finite dimensional vector space $\boldsymbol{V}$ is isomorphic to $\mathbb{C}^n$ or $\mathbb{R}^n$, with $n = \dim(\boldsymbol{V})$ because *ANY* basis has the same number of elements. Consequently it is enough to describe linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or $\mathbb{C}^n$ to $\mathbb{C}^m$) by real resp. complex $m \times n$-matrices. Thinking of a function oveer $\mathbb{R}$ is a "continuous collection of point values" $f(x), x \in \mathbb{R}$, it is natural to consider (linear) *INTEGRAL operators* of the form

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad x, y \in \mathbb{R}^d, \tag{4}$$

but is quickly clear that even some simple operators cannot be described in this way (e.g. multiplication operator, even the identity operator).

Kernel Thm., more

On the other hand, one may expect, that it is possible to regain the integral kernel $K(x,y)$ from the operator in a similar way as in the matrix case.
Recall that the k-th column of the matrix associated with T (given a basis) is the coordinate vector of $T(\mathbf{e}_k)$ in the target space.
Hence the expected "rule" to find the kernel reads:

$$K(x,y) = T(\delta_y)(x) = \delta_x(T(\delta_y)) \qquad (5)$$

but there is only a chance if δ_y is in the domain of the operator and the result, i.e. $T(\delta_y)$ is a continuous function, which can be evaluated pointwise at x.

The Hilbert Schmidt Version

There are two ways out of this problem

- restrict the class of operators
- enlarge the class of possible kernels

The first one is a classical result, i.e. the characterization of the class $\mathcal{HS}$ of Hilbert Schmidt operators.

Theorem

A linear operator T on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ is a Hilbert-Schmidt operator, i.e. is a compact operator with the sequence of singular values in ℓ^2 if and only if it is an integral operator of the form (4) with $K \in \boldsymbol{L}^2(\mathbb{R}^d \times \mathbb{R}^d)$. In fact, we have a unitary mapping $T \rightarrow K(x,y)$, where $\mathcal{HS}$ is endowed with the Hilbert-Schmidt scalar product $\langle T, S\rangle_{\mathcal{HS}} := \mathrm{trace}(T \circ S^*)$.

Fourier Standard Spaces of Operator Kernels I

The Kernel Theorem for the Banach Gelfand Triple allows to identify many spaces of operators with the corresponding Banach spaces of *integral kernels*. **The unitary mapping between $\boldsymbol{L}^2(\mathbb{R}^{2d})$ and $\mathcal{HS}$, the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H} = \left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, $K \mapsto T_K$:**

$$T_K(f)(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad x, y \in \mathbb{R}^d,$$

extends to a *Banach Gelfand Triple isomorphism* **between the $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ and the operator Banach Gelfand Triple $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$.** For $K \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ one has

$$K(x,y) = T(\delta_y)(x) = \delta_x(T(\delta_y)).$$

(in analogy to the matrix case: $a_{k,l} = \langle T(\mathbf{e}_l), \mathbf{e}_k \rangle$.

Fourier Standard Spaces of Operator Kernels II

Theorem

For any pair $(\mathbf{B}^1, \mathbf{B}^2)$, *with* $\mathbf{B}^1$ *minimal, the set*

$$\mathcal{K}(\mathbf{B}^1, \mathbf{B}^2) := \{\sigma_T \in \mathbf{S}_0'(G \times G),\ \ T \in \mathbf{L}(\mathbf{B}^1, \mathbf{B}^2)\}$$

of all kernels corresponding to bounded linear operators from $(\mathbf{B}^1, \|\cdot\|^{(1)})$ *to* $(\mathbf{B}^2, \|\cdot\|^{(2)})$ *is again a standard space on* $G \times G$, *endowed with the operator norm* $|||T|||_{\mathbf{B}^1 \to \mathbf{B}^2}$.

If $(\mathbf{B}^2, \|\cdot\|^{(2)})$ is a dual space of another (minimal) space $(\mathbf{B}^3, \|\cdot\|^{(3)})$ it is not difficult to identify the set of kernels with the dual of the *projective tensor product* $\mathbf{B}^1 \widehat{\otimes} \mathbf{B}^3$, which is a well defined subspace of $\mathbf{S}_0'(\mathbb{R}^d) \widehat{\otimes} \mathbf{S}_0'(\mathbb{R}^d) \subset \mathbf{S}_0'(\mathbb{R}^{2d})$, thus avoiding the abstract definition of tensor products of Banach spaces using certain norms and *completions*.

frametitle: bibliography I

P. Dimovski, S. Pilipovic, B. Prangoski, and J. Vindas.
Translation–modulation invariant Banach spaces of ultradistributions.
J. Fourier Anal. Appl., 25(3):819–841, 2019.

H. G. Feichtinger.
A compactness criterion for translation invariant Banach spaces of functions.
Analysis Mathematica, 8:165–172, 1982.

H. G. Feichtinger.
Compactness in translation invariant Banach spaces of distributions and compact multipliers.
J. Math. Anal. Appl., 102:289–327, 1984.

H. G. Feichtinger.
Banach convolution algebras of Wiener type.
In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of **Colloq. Math. Soc. Janos Bolyai**, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados. edition, 1983.

frametitle: bibliography II

H. G. Feichtinger and K. Gröchenig.
Banach spaces related to integrable group representations and their atomic decompositions, I.
J. Funct. Anal., 86(2):307–340, 1989.

H. G. Feichtinger and M. S. Jakobsen.
The inner kernel theorem for a certain Segal algebra.
2018.

H. G. Feichtinger and G. Zimmermann.
A Banach space of test functions for Gabor analysis.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170. Birkhäuser Boston, 1998.

H. Reiter and J. D. Stegeman.
Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.
Clarendon Press, Oxford, 2000.

Thanks and LINKS

More at www.nuhag.eu

All relevant papers are downloadable from

www.nuhag.eu/bibtex

This talk and many other talks on related subjects can be downloaded from

www.nuhag.eu/talks

User: visitor, PWD: nuhagtalks

Thank you for your attention!