# Sincov-type functional inequalities and generalized metrics

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# The outline of the talk

# Additive Sincov's equation

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Denote $f = T(\cdot, z_0)$ to get

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Conversely, for arbitrary function $f\colon X \to G$ map $T$ given as above solves (1).

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The general solution is given by $S = 0$ on $X \times X$ or

$$S(x,y) = \frac{f(x)}{f(y)}, \quad x,y \in X,$$

where $f\colon X \to \mathbb{R} \setminus \{0\}$ is an arbitrary function (see D. Gronau [2, Theorem]).

# First Sincov's inequality

We will study the inequality:

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$G$ solves (2) if and only if $G^*$ solves (2).

$$\textbf{(2)} \qquad G(x,z) \leq G(x,y)\cdot G(y,z)$$

### Example

For a non-void set $X$ every $G\colon X\times X\to(-\infty,0]$ and every $G\colon X\times X\to[c,c^2]$ with some $c\geq 1$ satisfies (2).

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### Proposition

*Assume that $X$ is a non-void set and $G\colon X\times X\to(0,+\infty)$ is a bounded solution of (2). Then there exists some $c\ge 1$ such that $G(X\times X)\subseteq[1/c,c]$.*

$$\textbf{(2)} \qquad G(x,z) \leq G(x,y) \cdot G(y,z)$$

### Proposition

*Assume that $X$ is a topological connected space and $G\colon X \times X \to \mathbb{R}$ is a continuous solution of (2). If $G$ attains a non-positive value, then $G$ is non-positive on $X \times X$.*

**(2)** $$G(x,z) \leq G(x,y) \cdot G(y,z)$$

### Example

Let $A\colon \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function with connected graph.

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Let $A\colon \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function with connected graph. Take $X = \mathbb{R}$ and

$$G_1(a,b) = \exp(A(a) - A(b)), \quad a, b \in \mathbb{R}.$$

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$G_1$ is a discontinuous solution of (2) with all sections having the Darboux property.

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## Example

Define $X = \{(x, A(x)) : x \in \mathbb{R}\}$ and

$$G_2((a, A(a)), (b, A(b))) = \exp(A(a) - A(b)), \quad a, b \in \mathbb{R}.$$

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### Example

Let $X$ be a disconnected topological space. Define $G_3(a,b)$ as being equal to 1 whenever $a, b$ lies in the same connected component of $X$ and $-1$ elsewhere.

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**Example**

Let $X$ be a disconnected topological space. Define $G_3(a, b)$ as being equal to 1 whenever $a, b$ lies in the same connected component of $X$ and $-1$ elsewhere. $G_3$ is a continuous solution of (2).

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We will denote the diagonal of the product $X \times X$ as

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**Theorem**

*Assume that $X$ is a separable topological space, $(x_0, y_0) \in X \times X$ and $G\colon X \times X \to (0, +\infty)$ is a solution of (2) such that $G$ is continuous and equal to $1$ at every point of $\Delta$.*

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## Theorem

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$$\mathcal{G}(G) := \left\{ f\colon X \to (0,+\infty) : \forall_{x,y\in X} \frac{f(x)}{f(y)} \leq G(x,y) \right\}.$$

$$\textbf{(2)} \qquad G(x,z) \leq G(x,y) \cdot G(y,z)$$

## Corollary

*Assume that $X$ is a separable topological space and $G\colon X \times X \to (0, +\infty)$ is a solution of (2) such that $G$ is continuous and equal to 1 at every point of $\Delta$.*

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## Corollary

*Assume that $X$ is a separable topological space and $G\colon X\times X\to(0,+\infty)$ is a solution of (2) such that $G$ is continuous and equal to 1 at every point of $\Delta$. Then*

$$G(a,b)=\sup\left\{\frac{f(a)}{f(b)} : f\in\mathcal{G}(G)\right\},\quad a,b\in X.$$

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*Conversely, for an arbitrary family $\mathcal{G}$ of positive functions on $X$ every mapping $G\colon X \times X \to (0,+\infty)$ defined as above solves Sincov's equation and is equal to 1 on $\Delta$.*

# Second Sincov's inequality

The reverse inequality to (2), i.e. the inequality

$$F(x,z) \geq F(x,y) \cdot F(y,z), \quad x,y,z \in X \tag{3}$$

is not fully symmetric to (2).

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$$F(x,z) \geq F(x,y) \cdot F(y,z), \quad x,y,z \in X \tag{3}$$

is not fully symmetric to (2).
Introduce a class of functions associated with a map $F\colon X \times X \to (0,+\infty)$.

$$\mathcal{F}(F) = \left\{ f\colon X \to (0,+\infty) : \forall_{x,y \in X} \frac{f(x)}{f(y)} \geq F(x,y) \right\}.$$

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## Corollary

*Assume that* $X$ *is a separable topological space and* $F\colon X \times X \to (0,+\infty)$ *is a solution of* (3) *which is continuous and equal to* $1$ *at every point of* $\Delta$.

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$$F(a,b) = \inf\left\{\frac{f(a)}{f(b)} : f \in \mathcal{F}(F)\right\}, \quad a, b \in X.$$

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$$\mathbf{(3)}\qquad F(x,z) \geq F(x,y)\cdot F(y,z)$$

### Lemma

*Assume that $X$ is a topological space and $F\colon X\times X\to[0,+\infty)$ is a continuous solution of* (3).

**(3)** $F(x,z) \geq F(x,y) \cdot F(y,z)$

## Lemma

*Assume that $X$ is a topological space and $F\colon X \times X \to [0,+\infty)$ is a continuous solution of (3). Then the set*

$$Z = \{(x,y) \in X \times X : F(x,y) = 0\}$$

*of zeros of $F$ is either empty, or for every point $(a,b) \in Z$, it contains either at least one of the sets $\{a\} \times X$ or $X \times \{b\}$, or a set of the form $U_1 \times \{b\} \cup \{a\} \times U_2$, where $U_1, U_2 \subset X$ are open non-void sets.*

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Given a set ideal $\mathcal{I}$ of subsets of a set $X$ we define the product ideal $\mathcal{I} \otimes \mathcal{I}$ of subsets of $X \times X$ as the family of all sets $A \subseteq X \times X$ such that

$$\{x \in X : A[x] \notin \mathcal{I}\} \in \mathcal{I},$$

where

$$A[x] = \{y \in X : (x, y) \in A\}.$$

$$\textbf{(3)} \qquad F(x,z) \geq F(x,y) \cdot F(y,z)$$

## Corollary

*Assume that* $X$ *is a topological space,* $\mathcal{I} \subset 2^X$ *is a set ideal which does not contain open non-void sets and* $F\colon X \times X \to [0,+\infty)$ *is a continuous solution of* (3).

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## Corollary

*Assume that $X$ is a topological space, $\mathcal{I} \subset 2^X$ is a set ideal which does not contain open non-void sets and $F\colon X \times X \to [0,+\infty)$ is a continuous solution of (3). Then, at least one of the following possibilities holds true:*

(a) *the set $Z$ of zeros of $F$ is empty,*
(b) *$Z$ contains a set of the form $\{a\} \times X$ or $X \times \{b\}$,*
(c) *$Z$ is a large set with respect to the product ideal $\mathcal{I} \otimes \mathcal{I}$ on $X \times X$.*

# Additive Sincov's inequality

*Generalized metric space* or *Lawvere space* is a nonempty set $X$ together with a function $H\colon X \times X \to \mathbb{R}$, called a generalized metric, which is nonnegative, vanishes on $\Delta$ and satisfies the triangle inequality:

$$H(x,z) \le H(x,y) + H(y,z), \quad x,y,z \in X. \tag{4}$$

# Additive Sincov's inequality

For arbitrary $H\colon X \times X \to \mathbb{R}$ let us define

$$\mathcal{H}(H) = \{\varphi\colon X \to \mathbb{R} : \forall_{x,y \in X} \varphi(x) - \varphi(y) \le H(x,y)\}\,.$$

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## Corollary

*Assume that $X$ is a separable topological space and $H\colon X \times X \to \mathbb{R}$ is a solution of $(4)$ which is continuous and equal to $0$ at every point of $\Delta$.*

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## Corollary

*Assume that $X$ is a separable topological space and $H\colon X \times X \to \mathbb{R}$ is a solution of (4) which is continuous and equal to 0 at every point of $\Delta$. Then*

$$H(a,b) = \sup\left\{\varphi(a) - \varphi(b) : \varphi \in \mathcal{H}(H)\right\}, \quad a, b \in X.$$

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*Assume that $X$ is a separable topological space and $H\colon X \times X \to \mathbb{R}$ is a solution of (4) which is continuous and equal to 0 at every point of $\Delta$. Then*

$$H(a, b) = \sup \{\varphi(a) - \varphi(b) : \varphi \in \mathcal{H}(H)\}, \quad a, b \in X.$$

*Conversely, for an arbitrary family $\mathcal{H}$ of real functions on $X$ every mapping $H\colon X \times X \to \mathbb{R}$ defined as above solves (4) and is equal to 0 on $\Delta$.*

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*Under assumptions of the last corollary, there exist a quotient subspace $X_0$ of $X$ such that:*

(a) *family $\mathcal{H}(H)$ separates points of $X_0$,*

(b) *every $\varphi \in \mathcal{H}(H)$ satisfies a Lipschitz-type condition*

$$|\varphi(a) - \varphi(b)| \leq \frac{1}{2}[H(a,b) + H(b,a)], \quad a, b \in X,$$

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## Corollary

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(c) $H|_{X_0 \times X_0}(a,b) = 0$ *if and only if* $a = b$.

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# Thank you for your kind attention!