Local Wiener's Theorem and Almost Periodic Measures and Distributions

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Theorem (N.Wiener)

Let a function $g(t)=\sum_{n\in\mathbb{Z}} c_n e^{2\pi int}$, $\quad \sum_{n\in\mathbb{Z}} |c_n|<\infty$, *not vanish at each point of* $[0,1]$. *Then*

$$\frac{1}{g(t)}=\sum_{n\in\mathbb{Z}} d_n e^{2\pi int}, \quad \sum_{n\in\mathbb{Z}} |d_n|<\infty.$$

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$$W=\left\{f(x)=\sum_n c_n e^{2\pi i\langle x,\gamma_n\rangle}, \quad x\in\mathbb{R}^d, \quad \gamma_n\in\mathbb{R}^d, \quad \sum_n |c_n|<\infty\right\}.$$

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Proposition (F, 2017)

For any $f\in W$ *and* $\varepsilon>0$ *there is* $g_\varepsilon\in W$ *such that* $g_\varepsilon(x)=1/f(x)$ *if* $|f(x)|\geq\varepsilon$ *and* $g_\varepsilon(x)=0$ *if* $|f(x)|\leq\varepsilon/2$.

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$S(\mathbb{R}^d)$ is Schwartz' space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$ with finite norms for all $n,\ m \in \mathbb{N} \cup \{0\}$

$$N_{n,m}(f) = \sup_{x\in\mathbb{R}^d}\{(1+|x|^n)\max_{k_1+\dots+k_d\le m}|\partial_{x_1}^{k_1}\dots\partial_{x_d}^{k_d}f(x)|\}.$$

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A set Λ is **uniformly discrete** $\equiv$ $\inf\{|\lambda-\lambda'| :\ \lambda,\lambda'\in\Lambda,\ \lambda\ne\lambda'\} > 0.$

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Theorem (Y.Meyer-A.Cordoba-M.Kolountzakis)

Let $\mu = \sum_{k=1}^{N} s_k \sum_{\lambda \in \Lambda_k} \delta_\lambda$ *be a Radon measure with uniformly discrete support* $\Lambda = \bigcup_k \Lambda_k$, *such that its Fourier transform* $\hat{\mu}$ *be an atomic Radon measure with the property*

$$|\hat{\mu}|\{x : |x| < r\} = O(r^d) \qquad (r \to \infty).$$

Then there exists a finite number of full-rank lattices L_i *and* $c_i \in \mathbb{R}^d$ *such that* $\Lambda = \cup_{i=1}^{N}(L_i + c_i)$.

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The same is valid for a Radon measure $\mu = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda$ with uniformly discrete Λ under the condition $\inf_{\lambda\in\Lambda} |a_\lambda| > 0$ and the same property of $\hat{\mu}$.

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If μ is an atomic Radon measure on $\mathbb{R}^d$ such that $|\mu|\{x : |x| < r\} = O(r^d)$ as $r \to \infty$, and its spectrum $\Gamma = \overline{\mathrm{supp}\hat{\mu}}$ is uniformly discrete, then $\hat{\mu}$ is the Radon measure.

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If, in addition, $\inf_{\gamma\in\Gamma} |\hat{\mu}(\gamma)| > 0$*, then*

$$\mu = \sum_{k=1}^{N} \sum_{j=1}^{J_k} \exp\{2\pi i\langle x, a_{j,k}\rangle\}\mu_k,$$

where μ_k *are d-periodic atomic Radon measures with full-rank lattices* L_k *of periods,* $k = 1, \dots, N$*, and* $a_{j,k} \in \mathbb{R}^d$*.*

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where μ_k are d-periodic atomic Radon measures with full-rank lattices L_k of periods, $k = 1, \dots, N$, and $a_{j,k} \in \mathbb{R}^d$.
Also, $\hat{\mu}$ is a measure of the form

$$\hat{\mu} = \sum\nolimits_{k=1}^{N} F_k(y) \sum\nolimits_{\gamma\in\Gamma_k} \delta_\gamma,$$

where $\Gamma_k = \cup_{j=1}^{J_k}[L_k^* + a_{j,k}]$ and $F_k \in W$.

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if f is a limit of finite exponential sums $\sum c_\lambda \exp\{2\pi i\langle x,\lambda\rangle\}$ with $\lambda\in\Lambda_1,\ c_\lambda\in\mathbb{C}$, w.r.t the topology of uniform convergence on every compact subset of $\mathbb{R}^d$, then f is almost periodic in the sense of H.Bohr.

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Theorem (F 2019)

Let, in addition, $\hat{\mu}$ be an atomic measure. Then for every $\varepsilon > 0$ the set $\Lambda_\varepsilon = \{\lambda : \ |a_\lambda| > \varepsilon\}$ satisfies Kahane's property.

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The Poisson formula is the equality for $f \in S(\mathbb{R}^d)$

$$\sum\nolimits_{k\in\mathbb{Z}^d} f(k) = \sum\nolimits_{k\in\mathbb{Z}^d} \hat{f}(k) \quad \text{or} \quad \hat{\mu} = \mu \quad \text{for } \mu = \sum\nolimits_{k\in\mathbb{Z}^d} \delta_k.$$

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Let $\mu = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda,\ a_\lambda \in \mathbb{C},\ \hat{\mu} = \sum_{\gamma\in\Gamma} b_\gamma \delta_\gamma,\ b_\gamma > 0,$ *be measures with the uniformly discrete set* Λ *and with the discrete and closed set* Γ*.*

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$$\Lambda \subset \cup_{j=1}^{N}(L + c_j), \quad \Gamma \subset \cup_{k=1}^{N'}(L^* + d_k), \quad c_1, \dots, c_N,\ d_1, \dots, d_{N'} \in \mathbb{R}^d.$$

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"L is a full-rank lattice" means $L = A(\mathbb{Z}^d)$ with a nonsingular linear operator A in $\mathbb{R}^d$, $L^* = A^*(\mathbb{Z}^d)$ is the conjugate lattice.

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Proposition

Every tempered distribution $F \in S^*(\mathbb{R}^d)$ with closed discrete support $\Lambda \subset \mathbb{R}^d$ has the form

$$F = \sum_{\lambda \in \Lambda} \sum_{k_1 + \cdots + k_d \le m} p_{\lambda,k} \partial_{x_1}^{k_1} \dots \partial_{x_d}^{k_d} \delta_\lambda, \quad p_{\lambda,k} \in \mathbb{C}, \quad k = (k_1, \dots, k_d),$$

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where $m = \mathrm{ord} F < \infty$. If

$$\exists h < \infty,\ c > 0 : \quad |\lambda - \lambda'| > c(1 + |\lambda|)^{-h} \quad \forall \lambda, \lambda' \in \Lambda, \quad \lambda \neq \lambda',$$

then $\|F_\lambda\| := \max_k |p_{\lambda,k}| = O(|\lambda|^T)$ as $|\lambda| \to \infty$ with some $T < \infty$.

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Let F *be a tempered distribution with support* Λ *and spectrum* Γ *such that the differences* $\Lambda - \Lambda$ *and* $\Gamma - \Gamma$ *both be closed discrete sets and one of them be uniformly discrete. Then* Λ *and* Γ *satisfy the same assertions as in the previous theorem.*

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We say that a distribution $F \in S^*(\mathbb{R}^d)$ with closed discrete support Λ is **large** if $\inf_{\lambda \in \Lambda} \|F_\lambda\| > 0$, where $\|F_\lambda\| = \max_k |p_{\lambda,k}|$.

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A set $\Lambda \subset \mathbb{R}^d$ is called **relatively dense** if there is R such that every ball of radius R intersects with Λ.

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i) $\hat{F}^1$, $\hat{F}^2$ are both atomic measures such that

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then Λ_1, Λ_2 *are finite unions of translates of a single full-rank lattice.*

Almost Periodic Distributions

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For any set Γ without the above condition there is non-almost periodic $F \in S^*(\mathbb{R}^d)$ such that $\hat{F}$ is a measure with support in Γ.

Fourier Coefficients of Almost Periodic Distributions

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Let g be an almost periodic function on $\mathbb{R}^d$. Its Fourier coefficients are defined by the formula

$$a_g(\gamma) = \lim_{R\to\infty} \frac{1}{\omega_d R^d} \int_{|x|<R} g(x) \exp\{-2\pi i\langle x, \gamma\rangle\} dx,$$

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The definition does not depend on f. Clearly, the set $\{\gamma : a_F(\gamma) \neq 0\}$ is countable too.

Fourier Transform of Almost Periodic Measures

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Every almost periodic measure μ is translation bounded. Also, we have

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There is an almost periodic function g such that $\hat{g}$ is not a measure.

Almost Periodic Measures with "Large" Coefficients

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Let μ be an almost periodic atomic Radon measure on $\mathbb{R}^d$ such that $|\mu|\{x : |x| < r\} = O(r^d)$ as $r \to \infty$, the set $\Gamma = \{\gamma \in \mathbb{R}^d : a_\mu(\gamma) \neq 0\}$ be uniformly discrete, and $\inf_{\gamma\in\Gamma} : |a_\mu(\gamma)| > 0$.

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Then

$$\mu = \sum_{k=1}^{N} \sum_{j=1}^{J_k} \exp\{2\pi i \langle x, b_{j,k}\rangle\}\mu_k,$$

where μ_k are d-periodic atomic Radon measures with full-rank lattices L_k of periods, $k = 1, \dots, N$, and $b_{j,k} \in \mathbb{R}^d$.

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Corollary (F., 2018)

For any almost periodic distribution F we have $\mathrm{supp}\hat{F} = \overline{\{\gamma : a_F(\gamma) \neq 0\}}$.

Thanks for your attention!