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ON THE ZEROS OF POLYNOMIALS WITH CONSTRAINED COEFFICIENTS

Tamás Erdélyi

Texas A&M University

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Let $S \subset \mathbb{C}$. Let $\mathcal{P}_n(S)$ be the set of all algebraic polynomials of degree at most n with each of their coefficients in S. A polynomial P of the form

$$P(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad a_j \in \mathbb{C} \,, \tag{1.1}$$

is called conjugate-reciprocal if

$$\overline{a}_j = a_{n-j} \,, \qquad j = 0, 1, \dots , n \,, \tag{1.2}$$

Functions T of the form

$$T(t) = \alpha_0 + \sum_{j=1}^{n} (\alpha_j \cos(jt) + \beta_j \sin(jt)) \,,$$

$$\alpha_j, \beta_j \in \mathbb{R} \,,$$

are called real trigonometric polynomials of degree at most n. It is easy to see that any real trigonometric polynomial T of degree at most n can be written as $T(t) = P(e^{it})e^{-int}$, where P is a conjugate-reciprocal algebraic polynomial of the form

$$P(z) = \sum_{j=0}^{2n} a_j z^j \,, \qquad a_j \in \mathbb{C} \,. \tag{1.3}$$

Conversely, if P is conjugate-reciprocal algebraic polynomial of the form (1.3), then there are

$$\theta_j \in \mathbb{R}\,, \qquad j = 1, 2, \dots n\,,$$

such that T defined by

$$T(t) := P(e^{it})e^{-int} = a_n + \sum_{j=1}^{n} 2|a_{j+n}|\cos(jt + \theta_j)$$

is a real trigonometric polynomial of degree at most n.

A polynomial P of the form (1.1) is called self-reciprocal if

$$a_j = a_{n-j}\,, \qquad j = 0, 1, \dots, n\,. \tag{1.4}$$

If the polynomial P above is self-reciprocal, then

$$T(t) := P(e^{it})e^{-int} = a_n + \sum_{j=1}^{n} 2a_{j+n}\cos(jt)\,.$$

Associated with an algebraic polynomial P of the form (1.1) we introduce the numbers

$$\mathrm{NC}(P) := \left|\{j \in \{0,1,\dots,n\} : a_j \neq 0\}\right|.$$

Here, and in what follows $|A|$ denotes the number of elements of a finite set A. Let $\mathrm{NZ}(P)$ denote the number of real zeros (by counting multiplicities) of an algebraic polynomial P on the unit circle.

Associated with a trigonometric polynomial

$$T(t) = \sum_{j=0}^{n} a_j \cos(jt)$$

we introduce the numbers

$$\mathrm{NC}(T) := \left|\{j \in \{0,1,\dots,n\} : a_j \neq 0\}\right|.$$

Let $\mathrm{NZ}(T)$ denote the number of real zeros (by counting multiplicities) of a trigonometric polynomial T in a period (of length 2π).

Let $\mathrm{NZ}^*(T)$ denote the number of sign changes of a trigonometric polynomial T in a period (of length 2π).

Let $0 \leq n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the form

$$T(\theta) = \sum_{j=1}^{N} \cos(n_j \theta)$$

must have at least one real zero in a period. This is obvious if $n_1 \neq 0$, since then the integral of the sum on a period is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently large N it follows from Littlewood's Conjecture simply.

Littlewood's Conjecture was proved by Konyagin [27] and independently by McGehee, Pigno, and Smith [36] in 1981. See also [13, pages 285-288] for a book proof. It is not difficult to prove the statement in general even in the case $n_1 = 0$ without using Littlewood's Conjecture. One possible way is to use the identity

$$\sum_{j=1}^{n_N} T\left(\frac{(2j-1)\pi}{n_N}\right) = 0\,.$$

See [28], for example. Another way is to use Theorem 2 of [35]. So there is certainly no shortage of possible approaches to prove the starting observation of this paper even in the case $n_1 = 0$.

It seems likely that the number of zeros of the above sums in a period must tend to ∞ with N. In a private communication B. Conrey asked how fast the number of real zeros of the above sums in a period tends to ∞ as a function N.

In [12] the authors observed that for an odd prime p the Fekete polynomial

$$f_p(z) = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) z^k$$

(the coefficients are Legendre symbols) has $\sim \kappa_0 p$ zeros on the unit circle, where

$$0.500813 > \kappa_0 > 0.500668\,.$$

Conrey's question in general does not appear to be easy.

In his monograph 'Some Problems in Real and Complex Analysis [32, problem 22] (1968) Littlewood poses the following research problem, which appears to still be open: 'If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^{N} \cos(n_m\theta)$? Possibly $N-1$, or not much less. Here real zeros are counted in a period.

In fact no progress appears to have been made on this in the last half century. In a recent paper [4] we showed that this is false. There exist cosine polynomials $\sum_{m=1}^{N} \cos(n_m\theta)$ with the n_m integral and all different so that the number of its real zeros in a period is

$$O(N^{9/10}(\log N)^{1/5})$$

(here the frequencies $n_m = n_m(N)$ may vary with N). However, there are reasons to believe that a cosine polynomial $\sum_{m=1}^{N} \cos(n_m\theta)$ always has many zeros in a period.

Let

$$\mathcal{L}_n := \left\{ P : \; P(z) = \sum_{j=0}^{n} a_j z^j \,, \;\; a_j \in \{-1, 1\} \right\} .$$

Elements of $\mathcal{L}_n$ are often called Littlewood polynomials of degree n. Let $\mathcal{K}_n$ be the set of all polynomials P of the form

$$P(z) = \sum_{j=0}^{n} a_j z^j \,, \;\; a_j \in \mathbb{C},$$

$$|a_0| = |a_n| = 1, \;\; |a_j| \leq 1 \,.$$

Observe that $\mathcal{L}_n \subset \mathcal{K}_n$. In [10] we proved that any polynomial $P \in \mathcal{K}_n$ has at least $8n^{1/2} \log n$ zeros in any open disk centered at a point on the unit circle with radius $33n^{-1/2} \log n$. Thus polynomials in $\mathcal{K}_n$ have quite a few zeros near the unit circle. One may naturally ask how many unimodular roots a polynomial in $\mathcal{K}_n$ can have.

Mercer [36] proved that if a Littlewood polynomial $P \in \mathcal{L}_n$ of the form (1.1) is skew reciprocal, that is,

$$a_j = (-1)^j a_{n-j}\,, \qquad j = 0, 1, \dots, n\,,$$

then it has no zeros on the unit circle. However, by using different elementary methods it was observed in both [17] and [36] that if a Littlewood polynomial P of the form (1.1) is self-reciprocal, that is,

$$a_j = a_{n-j}\,, \qquad j = 0, 1, \dots, n\,, \quad n \geq 1\,,$$

then it has at least one zero on the unit circle.

Mukunda [37] improved this by showing that every self-reciprocal Littlewood polynomial of odd degree has at least 3 zeros on the unit circle.

Drungilas [15] proved that every self-reciprocal Littlewood polynomial of odd degree $n \geq 7$ has at least 5 zeros on the unit circle and every self-reciprocal Littlewood polynomial of even degree $n \geq 14$ has at least 4 zeros on the unit circle.

In [7] we proved that the average number of zeros of self-reciprocal Littlewood polynomials of degree n is at least $n/4$.

However, it is much harder to give decent lower bounds for the quantities

$$\mathrm{NZ}_n := \min_P \mathrm{NZ}(P)\,,$$

where $\mathrm{NZ}(P)$ denotes the number of zeros of a polynomial P lying on the unit circle and the minimum is taken for all self-reciprocal Littlewood polynomials $P \in \mathcal{L}_n$. It has been conjectured for a long time that $\lim_{n\to\infty} \mathrm{NZ}_n = \infty$. In [15] we showed that $\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty$ whenever $P_n \in \mathcal{L}_n$ is self-reciprocal and

$$\lim_{n\to\infty} |P_n(1)| = \infty\,.$$

This follows as a consequence of a more general result, see Corollary 2.3 in [15], stated as Corollary 1.5 here, in which the coefficients of the self-reciprocal polynomials P_n of degree at most n belong to a fixed finite set of real numbers.

In [6] we proved the following result.

Theorem 1.1. *If the set* $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ *is finite, the set* $\{j \in \mathbb{N} : a_j \neq 0\}$ *is infinite, the sequence* (a_j) *is not eventually periodic, and*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt)\,,$$

then $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty\,.$

In [6] Theorem 1.1 is stated without the assumption that the sequence (a_j) is not eventually periodic. However, as the following example shows, Lemma 3.4 in [6], dealing with the case of eventually periodic sequences (a_j), is incorrect.

Let

$$\begin{aligned} T_{4n+1}(t) :=& \cos t + \cos((4n+1)t) \\ &+ \sum_{k=0}^{n-1} (\cos((4k+1)t) - \cos((4k+3)t)) \\ =& \frac{1+\cos((4n+2)t)}{2\cos t} + \cos t\,. \end{aligned}$$

It is easy to see that $T_{4n+1}(t) \neq 0$ on

$$[-\pi, \pi] \setminus \{-\pi/2, \pi/2\}$$

and the zeros of T_{4n+1} at $-\pi/2$ and $\pi/2$ are simple. Hence T_{4n+1} has only two (simple) zeros in a period. So the conclusion of Theorem 1.1 above is false for the sequence (a_j) with

$$a_0 := 0, \;\; a_1 := 2, \;\; a_3 := -1, \;\; a_{2k} := 0,$$

$$a_{4k+1} := 1, \;\; a_{4k+3} := -1, \;\; k = 1, 2, \ldots\,.$$

Nevertheless, Theorem 1.1 can be saved even in the case of eventually periodic sequences (a_j) if we assume that $a_j \neq 0$ for all sufficiently large j. See Lemma 3.11 in [34] where Theorem 1.1 in [6] is corrected as

Theorem 1.2. *If the set* $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ *is finite,* $a_j \neq 0$ *for all sufficiently large* j*, and*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt)\,,$$

then $\lim_{n\to\infty} \mathrm{NZ}(T_n) = \infty\,.$

Note that in the above theorems the coefficients of T_n come from a fixed (a_j). It was expected that the conclusion of Theorem 1.2 remains true even if the coefficients of T_n do not come from the same sequence, that is,

$$T_n(t) = \sum_{j=0}^{n} a_{j,n} \cos(jt)\,,$$

where the set

$$S := \{a_{j,n} : j \in \{0, 1, \ldots, n\},\, n \in \mathbb{N}\} \subset \mathbb{R}$$

is finite and

$$\lim_{n\to\infty} |\{j \in \{0, 1, \ldots, n\},\, a_{j,n} \neq 0\}| = \infty\,.$$

Associated with an algebraic polynomial

$$P(z) = \sum_{j=0}^{n} a_{j,n} z^j \,, \qquad a_{j,n} \in \mathbb{C} \,,$$

let

$$\mathrm{NC}_k(P)$$

$$:= \left| \left\{ u : 0 \leq u \leq n-k+1, \ \sum_{j=u}^{u+k-1} a_{j,n} \neq 0 \right\} \right| .$$

In [15] we proved the following results.

Theorem 1.3. *If $S \subset \mathbb{R}$ is finite, $P_{2n} \in \mathcal{P}_{2n}(S)$ are self-reciprocal polynomials,*

$$T_n(t) := P_{2n}(e^{it})e^{-int} \,,$$

and

$$\lim_{n \to \infty} \mathrm{NC}_k(P_{2n}) = \infty$$

for every $k \in \mathbb{N}$, then

$$\lim_{n \to \infty} \mathrm{NZ}(P_{2n}) = \lim_{n \to \infty} \mathrm{NZ}(T_n) = \infty \,.$$

Corollary 1.4. *If $S \subset \mathbb{R}$ is finite, $P_n \in \mathcal{P}_n(S)$ are self-reciprocal polynomials, and*

$$\lim_{n\to\infty} |P_n(1)| = \infty\,,$$

then

$$\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty\,.$$

Corollary 1.5. *Suppose the finite set $S \subset \mathbb{R}$ has the property that*

$$s_1 + s_2 + \cdots + s_k = 0\,, \quad s_1, s_2, \ldots, s_k \in S\,,$$

implies

$$s_1 = s_2 = \cdots = s_k = 0\,,$$

that is, any sum of nonzero elements of S is different from 0. If $P_n \in \mathcal{P}_n(S)$ are self-reciprocal polynomials and

$$\lim_{n\to\infty} \mathrm{NC}(P_n) = \infty\,,$$

then

$$\lim_{n\to\infty} \mathrm{NZ}(P_n) = \infty\,.$$

J. Sahasrabudhe [41] examined the case when $S \subset \mathbb{Z}$ is finite. Exploiting the assumption that the coefficients are integer he proved that for any finite set $S \subset \mathbb{Z}$ a self-reciprocal polynomial $P \in \mathcal{P}_{2n}(S)$ has at least

$$c\,(\log\log\log|P(1)|)^{1/2-\varepsilon} - 1$$

zeros on the unit circle of $\mathbb{C}$ with a constant $c > 0$ depending only on

$$M = M(S) := \max\{|z| : z \in S\}$$

and $\varepsilon > 0$.

Let $\phi(n)$ denote the Euler's totient function defined as the number of integers $0 < k \le n$ that are relative prime to n. In an earlier version of his paper Saharabudhe [41] used the trivial estimate $\phi(n) \ge \sqrt{n}$ for $n \ge 3$ and he proved his result with the exponent $1/4 - \varepsilon$ rather than $1/2 - \varepsilon$. Using the nontrivial estimate $\phi(n) \ge n/8(\log\log n)$ [1] for all $n > 3$ allowed him to prove his result with $1/2 - \varepsilon$.

In the papers [6], [15], [41] the already mentioned Littlewood Conjecture, proved by Konyagin [27] and independently by McGehee, Pigno, and B. Smith [34], plays a key role, and we rely on it heavily in the proof of the main results of this paper as well. This states the following.

Theorem 1.6. *There exists an absolute constant* $c > 0$ *such that*

$$\int_0^{2\pi} \Big| \sum_{j=1}^{m} a_j e^{i\lambda_j t} \Big| \, dt \geq c\gamma \log m$$

whenever $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are distinct integers and* $a_1, a_2, \ldots, a_m$ *are complex numbers of modulus at least* $\gamma > 0$. *Here* $c = 1/30$ *is a suitable choice.*

This is an obvious consequence of the following result a book proof of which has been worked out by Lorentz and DeVore in [13, pages 285-288].

Theorem 1.7. *If* $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ *are integers and* $a_1, a_2, \ldots, a_m$ *are complex numbers, then*

$$\int_0^{2\pi} \Big| \sum_{j=1}^{m} a_j e^{i\lambda_j t} \Big| \, dt \geq \frac{1}{30} \sum_{j=1}^{m} \frac{|a_j|}{j} \, .$$

In [E-19e] we proved the following results. For the sake of brevity let

$$M = M(S) := \max\{|z| : z \in S\}\,.$$

Theorem 1.8. *If $S \subset \mathbb{Z}$ is finite, $P \in \mathcal{P}_{2n}(S)$ is a self-reciprocal polynomial,*

$$T(t) := P(e^{it})e^{-int}\,,$$

$|P(1)| \geq 16$, then

$$\mathrm{NZ}^*(T) \geq \left(\frac{c}{1+\log M}\right) \frac{\log\log\log|P(1)|}{\log\log\log\log|P(1)|} - 1$$

with an absolute constant $c > 0$.

Corollary 1.9. *If $S \subset \mathbb{Z}$ is finite, $P \in \mathcal{P}_n(S)$ is a self-reciprocal polynomial, $|P(1)| \geq 16$, then*

$$\mathrm{NZ}(P) \geq \left(\frac{c}{1+\log M}\right) \frac{\log\log\log|P(1)|}{\log\log\log\log|P(1)|} - 1$$

with an absolute constant $c > 0$.

This improves the exponent $1/2 - \varepsilon$ to $1 - \varepsilon$ in a recent breakthrough result [S-19a] by Sahasrabudhe.

We note that in both Sahasrabudhe's paper and this paper the assumption that the finite set S contains only integers is deeply exploited.

Our next result is an obvious consequence of Corollary 1.9.

Corollary 1.10. *If the set $S \subset \mathbb{Z}$ is finite,*

$$T(t) = \sum_{j=0}^{n} a_j \cos(jt)\,, \qquad a_j \in S\,,$$

$|T(0)| \geq 16$, *then*

$$\mathrm{NZ}^*(T) \geq \left(\frac{c}{1+\log M}\right) \frac{\log\log\log|T(0)|}{\log\log\log\log|T(0)|} - 1$$

with an absolute constant $c > 0$.

2. On the multiplicity of the zero at 1 of constrained coefficients

In [BEK-99] and [BEK-13] we examined a number of problems concerning polynomials with coefficients restricted in various ways. We were particularly interested in how small such polynomials can be on $[0, 1]$. For example, we proved that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$e^{-c_1\sqrt{n}} \leq \min_{0 \not\equiv Q \in \mathcal{F}_n} \left\{ \max_{x \in [0,1]} |Q(x)| \right\} \leq e^{-c_2\sqrt{n}}$$

for every $n \geq 2$, where $\mathcal{F}_n$ denotes the set of all polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$.

Littlewood considered minimization problems of this variety on the unit disk. His most famous, now solved, conjecture was that the L_1 norm of an element $f \in \mathcal{F}_n$ on the unit circle grows at least as fast as $c \log N$, where N is the number of non-zero coefficients in f and $c > 0$ is an absolute constant.

When the coefficients are required to be integers, the questions have a Diophantine nature and have been studied from a variety of points of view.

One key to the analysis is a study of the related problem of giving an upper bound for the multiplicity of the zero these restricted polynomials can have at 1. In [BEK-99] and [BEK-13] we answer this latter question precisely for the class of polynomials of the form

$$Q(x) = \sum_{j=0}^{n} a_j x^j \,,$$

$$|a_j| \leq 1 \,, \quad a_j \in \mathbb{C} \,, \quad j = 1, 2, \ldots, n \,,$$

with fixed $|a_0| \neq 0$.

Various forms of these questions have attracted considerable study, though rarely have precise answers been possible to give. Indeed, the classical, much studied, and presumably very difficult problem of Prouhet, Tarry, and Escott rephrases as a question of this variety. (Precisely: what is the maximal vanishing at 1 of a polynomial with integer coefficients with l_1 norm $2n$? It is conjectured to be n.)

For $n \in \mathbb{N}$, $L > 0$, and $p \geq 1$ let $\kappa_p(n, L)$ be the largest possible value of k for which there is a polynomial $Q \not\equiv 0$ of the form

$$Q(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad a_j \in \mathbb{C}\,,$$

$$|a_0| \geq L \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} ,$$

such that $(x-1)^k$ divides $Q(x)$.

For $n \in \mathbb{N}$ and $L > 0$ let $\kappa_\infty(n, L)$ be the largest possible value of k for which there is a polynomial $Q \not\equiv 0$ of the form

$$Q(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad a_j \in \mathbb{C}\,,$$

$$|a_0| \geq L \max_{1 \leq j \leq n} |a_j| \,,$$

such that $(x-1)^k$ divides $Q(x)$.

In [BEK-99] we proved that there is an absolute constant $c_3 > 0$ such that

$$\min\left\{\frac{1}{6}\sqrt{(n(1-\log L)} - 1\,, n\right\} \le \kappa_\infty(n,L)$$

$$\le \min\left\{c_3\sqrt{n(1-\log L)}\,, n\right\}$$

for every $n \in \mathbb{N}$ and $L \in (0,1]$. However, we were far from being able to establish the right result in the case of $L \ge 1$. In [BEK-13] we proved the right order of magnitude of $\kappa_\infty(n,L)$ and $\kappa_2(n,L)$ in the case of $L \ge 1$. Our results in [BEK-99] and [BEK-13] sharpen and generalize results of Schur [Sch-33], Amoroso [A-90], Bombieri and Vaaler [BV-87], and Hua [H-82] who gave versions of this result for polynomials with integer coefficients. Our results in [BEK-99] have turned out to be related to a number of recent papers from a rather wide range of research areas.

For $n \in \mathbb{N}$, $L > 0$, and $q \geq 1$ let $\mu_q(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L}\left(\sum_{j=1}^{n} |Q(j)|^q\right)^{1/q}.$$

For $n \in \mathbb{N}$ and $L > 0$ let $\mu_\infty(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \max_{1 \leq j \leq n} |Q(j)|\,.$$

It is a simple consequence of Hölder's inequality (see Lemma 3.6) that

$$\kappa_p(n, L) \leq \mu_q(n, L)$$

whenever $n \in \mathbb{N}$, $L > 0$, $1 \leq p, q \leq \infty$, and

$$1/p + 1/q = 1\,.$$

In [E-15a] we find the the size of $\kappa_p(n,L)$ and $\mu_q(n,L)$ for all $n \in \mathbb{N}$, $L > 0$, and $1 \leq p,q \leq \infty$. The result about $\mu_\infty(n,L)$ is due to Coppersmith and Rivlin, [CR-92], but our proof presented in [E-15a] is completely different and much shorter even in that special case.

Our results in [BEK-99] may be viewed as finding the size of $\kappa_\infty(n,L)$ and $\mu_1(n,L)$ for all $n \in \mathbb{N}$ and $L \in (0,1]$.

Our results in [BEK-13] may be viewed as finding the size of $\kappa_\infty(n,L)$, $\mu_1(n,L)$, $\kappa_2(n,L)$ and $\mu_2(n,L)$ for all $n \in \mathbb{N}$ and $L > 0$.

Theorem 2.1. *Let $p \in (1,\infty]$ and $q \in [1,\infty)$ satisfy $1/p+1/q=1$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\begin{aligned}\sqrt{n}(c_1 L)^{-q/2} - 1 \le \kappa_p(n,L) &\le \mu_q(n,L)\\ &\le \sqrt{n}(c_2 L)^{-q/2} + 2\end{aligned}$$

for every $n \in \mathbb{N}$ and $L > 1/2$, and

$$c_3 \min\left\{\sqrt{n(-\log L)}, n\right\} \le \kappa_p(n,L) \le \mu_q(n,L)$$

$$\le c_4 \min\left\{\sqrt{n(-\log L)}, n\right\} + 4$$

for every $n \in \mathbb{N}$ and $L \in (0,1/2]$. Here $c_1 := 1/53$, $c_2 := 40$, $c_3 := 2/7$, and $c_4 := 13$ are appropriate choices.

Theorem 2.2. *There are absolute constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$c_1\sqrt{n(1-L)} - 1 \le \kappa_1(n,L) \le \mu_\infty(n,L) \le c_2\sqrt{n(1-L)} + 1$$

for every $n \in \mathbb{N}$ *and* $L \in (1/2, 1]$, *and*

$$c_3 \min\left\{\sqrt{n(-\log L)}, n\right\} \le \kappa_1(n,L) \le \mu_\infty(n,L)$$

$$\le c_4 \min\left\{\sqrt{n(-\log L)}, n\right\} + 4$$

for every $n \in \mathbb{N}$ *and* $L \in (0, 1/2]$. *Note that* $\kappa_1(n,L) = \mu_\infty(n,L) = 0$ *for every* $n \in \mathbb{N}$ *and* $L > 1$. *Here* $c_1 := 1/5$, $c_2 := 1$, $c_3 := 2/7$, *and* $c_4 := 13$ *are appropriate choices.*

3. Remarks and Problems

A question we have not considered in [E-15a] is if there are examples of n, L, and p for which the values of $\kappa_p(n, L)$ are significantly smaller if the coefficients are required to be rational (perhaps together with other restrictions). The same question may be raised about $\mu_q(n, L)$. As the conditions on the coefficients of the polynomials in Theorems 2.1 and 2.2 are homogeneous, assuming rational coefficients and integer coefficients lead to the same results. Three special classes of interest are

$$\mathcal{F}_n := \left\{ Q : Q(z) = \sum_{j=0}^{n} a_j z^j \,, \;\; a_j \in \{-1, 0, 1\} \right\} ,$$

$$\mathcal{L}_n := \left\{ Q : Q(z) = \sum_{j=0}^{n} a_j z^j \,, \;\; a_j \in \{-1, 1\} \right\} ,$$

and

$$\mathcal{K}_n :=$$

$$:= \left\{ Q : Q(z) = \sum_{j=0}^{n} a_j z^j \,, \;\; a_j \in \mathbb{C}, \;\; |a_j| = 1 \right\} .$$

The following three problems arise naturally.

Problem 3.1. *How many zeros can a polynomial* $0 \not\equiv Q \in \mathcal{F}_n$ *have at* 1*?*

Problem 3.2. *How many zeros can a polynomial* $Q \in \mathcal{L}_n$ *have at* 1*?*

Problem 3.3. *How many zeros can a polynomial* $Q \in \mathcal{K}_n$ *have at* 1*?*

The case when $p = \infty$ and $L = 1$ in our Theorem 2.1 gives that every $0 \not\equiv Q \in \mathcal{F}_n$, every $Q \in \mathcal{L}_n$, and every $Q \in \mathcal{K}_n$ can have at most $cn^{1/2}$ zeros at 1 with an absolute constant $c > 0$, but one may expect better results by utilizing the additional pieces of information on their coefficients.

It was observed in [BEK-99] that for every integer $n \geq 2$ there is a $Q \in \mathcal{F}_n$ having at least $c(n/\log n)^{1/2}$ zeros at 1 with an absolute constant $c > 0$. This is a simple pigeon hole argument. However, as far as we know, closing the gap between $cn^{1/2}$ and $c(n/\log n)^{1/2}$ in Problem 3.1 is an open and most likely very difficult problem.

As far as Problem 3.2 is concerned, Boyd [B-97] showed that for $n \geq 3$ every $Q \in \mathcal{L}_n$ has at most

$$\frac{c(\log n)^2}{\log \log n} \tag{3.1}$$

zeros at 1, and this is the best known upper bound even today. Boyd's proof is very clever and up to an application of the Prime Number Theorem is completely elementary. It is reasonable to conjecture that for $n \geq 2$ every $Q \in \mathcal{L}_n$ has at most $c \log n$ zeros at 1. It is easy to see that for every integer $n \geq 2$ there are $Q_n \in \mathcal{L}_n$ with at least $c \log n$ zeros at 1 with an absolute constant $c > 0$.

As far as Problem 3.3 is concerned, one may suspect that for $n \geq 2$ every $Q \in \mathcal{K}_n$ has at most $c \log n$ zeros at 1. However, just to see if Boyd's bound (3.1) holds for every $Q \in \mathcal{K}_n$ seems quite challenging and beyond reach at the moment.

Problem 3.4. *How many zeros can a polynomial $P \in \mathcal{F}_n$ have at α if $|\alpha| \neq 1$ and $\alpha \neq 0$? Can it have as many as we want?*

Problem 3.5. *How many zeros can a polynomial $P \in \mathcal{L}_n$ have at α if $|\alpha| \neq 1$ and $\alpha \neq 0$? Can it have as many as we want want?*

The Mahler measure

$$M_0(P) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{it})|\,dt\right)$$

is defined for bounded measurable functions P defined on the unit circle. It is well known that

$$M_0(P) := \lim_{q\to 0+} M_q(P)\,,$$

where, for $q > 0$,

$$M_q(P) := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{it})\right|^q\,dt\right)^{1/q}\,.$$

It is a simple consequence of the Jensen formula that

$$M_0(P) = |c|\prod_{k=1}^{n}\max\{1,|z_k|\}$$

for every polynomial of the form

$$P(z) = c\prod_{k=1}^{n}(z - z_k)\,, \qquad c, z_k \in \mathbb{C}\,.$$

Lehmer's conjecture is a problem in number theory raised by Derrick Henry Lehmer. The conjecture asserts that there is an absolute constant $\mu > 1$ such that for every polynomial P with integer coefficients satisfying $P(0) \neq 0$ we have either $M_0(P) = 1$ (that is, P is monic and has all its zeros on the unit circle) or $M_0(P) \geq \mu$.

The smallest known Mahler measure greater than 1 is taken for the "Lehmer's polynomial"

$$P(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

for which

$$M_0(P) = 1.176280818\ldots .$$

It is widely believed that this example represents the true minimal value: that is,

$$\mu = 1.176280818\ldots$$

in Lehmer's conjecture.

It is a simple counting argument (Bombieri and Vaaler) to show that if $k \geq 2$ is an integer, the monic polynomial Q has only integer coefficients, and $M_0(Q) < k$, then there is a polynomial P with integer coefficients in $[-k+1, k-1]$ such that Q divides P.

In particular, if the monic polynomial Q has only integer coefficients, and $M_0(Q) < 2$, then there is a polynomial $P \in \mathcal{F}_n$ such that Q divides P.

Observe that for

$$Q(z) = (z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1)^4$$

we have $M_0(Q) = (1.176280818\ldots)^4 < 2$ hence there is a polynomial $P \in \mathcal{F}_n$ such that Q divides P.