Simultaneous approximation by Bernstein polynomials and their integer forms

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AnMath 2019
August 12–17, 2019 / Budapest, Hungary

Supported by grant DN 02/14 of the Fund for Scientific Research of the Bulgarian Ministry of Education and Science

The Bernstein operator

$$(1)\quad B_nf(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$$

$$\lim_{n\to\infty} \|B_nf - f\| = 0, \quad f \in C[0,1]$$

The Bernstein operator

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$$\lim_{n\to\infty} \|B_n f - f\| = 0, \quad f \in C[0,1]$$

$$\lim_{n\to\infty} \|(B_n f)^{(s)} - f^{(s)}\| = 0, \quad f \in C^s[0,1]$$

An error characterization

(2) $$\|B_n f - f\| \sim K(f, n^{-1})$$

(3) $$K(f,t) := \inf_{g \in C^2[0,1]} \left\{ \|f - g\| + t\|\varphi^2 g''\| \right\}$$

$$\varphi(x) := \sqrt{x(1-x)}$$

The converse inequality: Knoop and Zhou (1994); Totik (1994)

First derivative

H. Berens and Y. Xu (1990), Gonska and Zhou (1995):

(4) $$\|(B_n f)' - f'\| \sim K_1(f', n^{-1})$$

(5) $$K_1(f', t) := \inf_{g \in C^3[0,1]} \left\{ \|f' - g'\| + t\|(\varphi^2 g'')'\| \right\}$$

(6) $$K(f, t) := \inf_{g \in C^2[0,1]} \left\{ \|f - g\| + t\|\varphi^2 g''\| \right\}$$

$$\varphi(x) := \sqrt{x(1-x)}$$

The direct estimate

Theorem

Let: $1 < p \leq \infty, \quad w(x) := x^{\gamma_0}(1-x)^{\gamma_1},$

$$-1/p < \gamma_0, \gamma_1 < s - 1/p \quad \text{if} \quad 1 < p < \infty$$
$$0 \leq \gamma_0, \gamma_1 < s \quad \text{if} \quad p = \infty.$$

Then $\forall f \in C[0,1] : \; f \in AC_{loc}^{s-1}(0,1), \quad wf^{(s)} \in L_p[0,1]$ and $\forall n \in \mathbb{N} \quad \Rightarrow$

(7) $$\|w((B_n f)^{(s)} - f^{(s)})\|_p \leq c\, K_s(f^{(s)}, n^{-1})_{w,p}.$$

$$K_s(f^{(s)}, t)_{w,p} := \inf_{g \in C^{s+2}[0,1]} \left\{ \|w(f^{(s)} - g^{(s)})\|_p + t\|w(\varphi^2 g'')^{(s)}\|_p \right\}$$

A strong converse inequality

Theorem

Let: $1 < p \leq \infty, \quad w(x) := x^{\gamma_0}(1-x)^{\gamma_1}$,

$$-1/p < \gamma_0, \gamma_1 < s - 1/p \quad \text{if} \quad 1 < p < \infty$$

$$0 \leq \gamma_0, \gamma_1 < s \quad \text{if} \quad p = \infty.$$

Then $\forall f \in C[0,1]: \ f \in AC_{loc}^{s-1}(0,1), \quad wf^{(s)} \in L_p[0,1]$ and $\forall n \in \mathbb{N} \quad \Rightarrow$

$$K_s(f^{(s)}, n^{-1})_{w,p} \leq c \left(\|w((B_n f)^{(s)} - f^{(s)})\|_p + \|w((B_{Rn} f)^{(s)} - f^{(s)})\|_p \right).$$

R – independent of f and n

A simpler form of the characterization I

$$1 < p \le \infty, \quad -1/p < \gamma_0, \gamma_1 < s - 1/p$$

(8) $$K_s(f, n^{-1})_{w,p} \sim \begin{cases} \omega_\varphi^2(f, n^{-1/2})_{w,p} + \omega_1(f, n^{-1})_{w,p}, & s = 1 \\ \omega_\varphi^2(f, n^{-1/2})_{w,p} + \dfrac{1}{n}\|wf\|_p, & s \ge 2 \end{cases}$$

$p = \infty$ and $w = 1$

(9)

$$K_s(f, n^{-1})_{1,\infty} \sim \begin{cases} \omega_\varphi^2(f, n^{-1/2})_\infty + \omega_1(f, n^{-1})_\infty, & s = 1 \\ \omega_\varphi^2(f, n^{-1/2})_\infty + \omega_1(f, n^{-1})_\infty + \dfrac{1}{n}\|f\|_\infty, & s \ge 2 \end{cases}$$

A simpler form of the characterization II

The Ditzian-Totik modulus, 1987:

$$\omega_{\varphi}^{2}(f,t) := \sup_{0<h\leq t} \|\bar{\Delta}_{h\varphi}^{2} f\|,$$

where

$$(10) \quad \bar{\Delta}_{h\varphi(x)}^{2} f(x) := \begin{cases} f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x)), & x \pm h\varphi(x) \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Direct estimates—the unweighted case

$$\|(B_nf)^{(s)} - f^{(s)}\| \le c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}), & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\|f^{(s)}\|, & s \ge 2 \end{cases}$$

Integer Bernstein polynomials

(11) $$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

An integer form, Kantorovich, 1931

(12) $$\widetilde{B}_n(f)(x) := \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] x^k (1-x)^{n-k}$$

The approximation rate

$f \in C[0,1], \quad f(0), f(1) \in \mathbb{Z}$:

(13)

$$c^{-1}\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) \le \|\widetilde{B}_n(f) - f\| + \frac{1}{n} \le c\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right)$$

Simultaneous approximation I

Let:

- $f \in C^s[0,1]$;
- $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$;
- $f^{(i)}(0) = f^{(i)}(1) = 0, \quad i = 2, \dots, s$;
- There exist $n_0 \in \mathbb{N}$ such that

$$f\left(\frac{k}{n}\right) \geq f(0) + \frac{k}{n} f'(0), \quad k = 1, \dots, s, \ n \geq n_0,$$
$$f\left(\frac{k}{n}\right) \geq f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s, \dots, n-1, \ n \geq n_0.$$

Simultaneous approximation II

Then for $n \geq n_0$ there holds

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\|$$
$$\leq c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}) + \dfrac{1}{n}, & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\|f^{(s)}\| + \dfrac{1}{n}, & s \geq 2. \end{cases}$$

The constant c is independent of f and n.

Another integer Bernstein polynomial

$$(14) \qquad \widehat{B}_n(f)(x) := \sum_{k=0}^{n} \left\langle f\left(\frac{k}{n}\right)\binom{n}{k}\right\rangle x^k(1-x)^{n-k}$$

$\langle\alpha\rangle$ — the nearest integer; tie-breaking rules:

- Round half up/down;
- Round half towards/away from zero;
- Round half to even/odd;
- Random half-rounding.

(15)

$$c^{-1}\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right) \le \|\widehat{B}_n(f) - f\| + \frac{1}{n} \le c\left(\omega_\varphi^2(f, n^{-1/2}) + \frac{1}{n}\right)$$

Simultaneous approximation by $\widehat{B}_n$

Let:

- $f \in C^s[0,1]$;
- $f(0), f(1), f'(0), f'(1) \in \mathbb{Z}$;
- $f^{(i)}(0) = f^{(i)}(1) = 0, \quad i = 2, \ldots, s.$

Then for $n \geq 1$ there holds

$$\|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| \leq c \begin{cases} \omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1}) + \dfrac{1}{n}, & s = 1, \\ \omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \dfrac{1}{n}\|f^{(s)}\| + \dfrac{1}{n}, & s \geq 2. \end{cases}$$

The constant c is independent of f and n.

Necessary conditions I

$$(16) \quad \lim_{n\to\infty} \|\widehat{B}_n(f) - f\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0$$

imply

- $f^{(i)}(0), f^{(i)}(1) \in \mathbb{Z}, \quad i = 0, \dots, s;$

Necessary conditions I

$$(16)\quad \lim_{n\to\infty} \|\widehat{B}_n(f) - f\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = 0$$

imply

- $f^{(i)}(0), f^{(i)}(1) \in \mathbb{Z}, \quad i = 0, \ldots, s;$
- $f^{(i)}(0) = f^{(i)}(1) = 0, \quad i = 2, \ldots, s.$

Necessary conditions II

$$(17) \quad \lim_{n\to\infty} \|\widetilde{B}_n(f) - f\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = 0$$

imply

- $f^{(i)}(0), f^{(i)}(1) \in \mathbb{Z}, \quad i = 0, \ldots, s;$

Necessary conditions II

$$(17)\quad \lim_{n\to\infty} \|\widetilde{B}_n(f) - f\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = 0$$

imply

- $f^{(i)}(0), f^{(i)}(1) \in \mathbb{Z}, \quad i = 0, \ldots, s;$
- $f^{(i)}(0) = f^{(i)}(1) = 0, \quad i = 2, \ldots, s;$

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$$(17)\quad \lim_{n\to\infty} \|\widetilde{B}_n(f) - f\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = 0$$

imply

- $f^{(i)}(0), f^{(i)}(1) \in \mathbb{Z}, \quad i = 0, \ldots, s;$
- $f^{(i)}(0) = f^{(i)}(1) = 0, \quad i = 2, \ldots, s;$
- there exists $n_0 \in \mathbb{N}$ such that

$$f\left(\frac{k}{n}\right) \geq f(0) + \frac{k}{n} f'(0), \quad k = 1, \ldots, s,\ n \geq n_0,$$
$$f\left(\frac{k}{n}\right) \geq f(1) - \left(1 - \frac{k}{n}\right) f'(1), \quad k = n-s, \ldots, n-1,\ n \geq n_0.$$

Converse estimates

Let:

- $f \in C^s[0,1]$ and $f(0), f(1) \in \mathbb{Z}$;
- $0 < \alpha < 1$ and

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}) \quad \text{or} \quad \|(\widehat{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha}).$$

Then

$$\omega_\varphi^2(f^{(s)}, h) = O(h^{2\alpha}) \quad \text{and} \quad \omega_1(f^{(s)}, h) = O(h^{\alpha}).$$

An equivalence relation

Under the assumptions for the direct estimate, we have

$$\|(\widetilde{B}_n(f))^{(s)} - f^{(s)}\| = O(n^{-\alpha})$$

$$\iff \quad \omega_\varphi^2(f^{(s)}, h) = O(h^{2\alpha}) \quad \text{and} \quad \omega_1(f^{(s)}, h) = O(h^\alpha)$$

$0 < \alpha < 1$

Kantorovich polynomials with integer coefficients I

$$(18)\quad K_n f(x) := \sum_{k=0}^{n} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, dt\, p_{n,k}(x),$$

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$$(19)\quad \widetilde{K}_n(f)(x) := \sum_{k=0}^{n} \left[(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, dt \binom{n}{k} \right] x^k (1-x)^{n-k}$$

Kantorovich polynomials with integer coefficients II

(20) $$K_n f(x) := (B_{n+1}F(x))', \quad F(x) := \int_0^x f(t)\,dt$$

(21)

$$\widetilde{K}_n(f)(x) := \left(\widetilde{B}_{n+1}(F)(x)\right)'$$

$$= \sum_{k=0}^{n} \left((k+1) \left[\int_0^{\frac{k+1}{n+1}} f(t)\,dt \binom{n+1}{k+1} \right] - (n-k+1) \left[\int_0^{\frac{k}{n+1}} f(t)\,dt \binom{n+1}{k} \right] \right) x^k (1-x)^{n-k}$$

Kantorovich polynomials with integer coefficients III

Assuming:

$$(22) \qquad f(0), f(1) \in \mathbb{Z}; \quad \int_0^1 f(t)\,dt \in \mathbb{Z};$$

$$(23) \qquad n\int_0^{1/n} f(t)\,dt \geq f(0); \quad n\int_{1-1/n}^1 f(t)\,dt \leq f(1).$$

Then

$$(24) \qquad \|\widetilde{K}_n(f) - f\| \leq c\left(\omega_\varphi^2(f, n^{-1/2}) + \omega_1(f, n^{-1}) + \frac{1}{n}\right).$$

THE END

Jackson-type estimates I

Theorem

Let: $1 < p \leq \infty$

$$-1/p < \gamma_0, \gamma_1 \leq s$$

$$s' := \max\{s, 2\}.$$

Then $\forall f \in C[0,1] : \ f \in AC_{loc}^{s+1}(0,1), \quad$ and $\quad \forall n \in \mathbb{N} \quad \Rightarrow$

$$\|w(B_n f - f)^{(s)}\|_p \leq \frac{c}{n}\left(\|wf^{(s')}\|_p + \|w\varphi^{2r} f^{(s+2)}\|_p\right).$$

Jackson-type estimates II

Theorem

Let $s' := \max\{s, 2\}$.

Then $\forall f \in C[0,1]: \ f \in AC_{loc}^{s+1}(0,1),$ and $\forall n \in \mathbb{N} \quad \Rightarrow$

$$\|(B_n f - f)^{(s)}\|_\infty \leq \frac{c}{n} \left(\|f^{(s')}\|_\infty + \|f^{(s+1)}\|_\infty + \|\varphi^{2r} f^{(s+2)}\|_\infty \right).$$

Kantorovich operators—the direct estimate

Theorem

Let: $1 < p \le \infty$

$$-1/p < \gamma_0, \gamma_1 < s + 1 - 1/p \quad \text{if} \quad 1 < p < \infty$$

$$0 \le \gamma_0, \gamma_1 < s + 1 \quad \text{if} \quad p = \infty.$$

Then $\forall f \in L_p[0,1]: \quad f \in AC_{loc}^{s-1}(0,1), \quad \text{and} \quad \forall n \in \mathbb{N} \quad \Rightarrow$

$$\|w(K_n f - f)^{(s)}\|_p \le c\, K_{s+1}(f^{(s)}, n^{-1})_{w,p}. \tag{25}$$

$$K_{s+1}(f^{(s)}, t)_{w,p} := \inf_{g \in C^{s+2}[0,1]} \left\{ \|w(f^{(s)} - g^{(s)})\|_p + t\|w(\varphi^2 g')^{(s+1)}\|_p \right\}$$

Kantorovich operators—a strong converse inequality

Theorem

Let: $1 < p \leq \infty$

$$-1/p < \gamma_0, \gamma_1 < s + 1 - 1/p \quad \text{if} \quad 1 < p < \infty$$

$$0 \leq \gamma_0, \gamma_1 < s + 1 \quad \text{if} \quad p = \infty.$$

Then $\forall f \in L_p[0,1]: \; f \in AC_{loc}^{s-1}(0,1), \quad wf^{(s)} \in L_p[0,1]$ and $\forall n \in \mathbb{N} \quad \Rightarrow$

$$K_{s+1}(f^{(s)}, n^{-1})_{w,p} \leq c \left(\|w(K_n f - f)^{(s)}\|_p + \|w(K_{Rn} f - f)^{(s)}\|_p \right).$$

R – independent of f and n

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