On some local properties of the conjugate function and the modulus of smoothness of fractional order

Ana Danelia

Department of Mathematics, Faculty of Exact and Natural Sciences,
Ivane Javakhishvili Tbilisi State University

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abstract

In the present report we study the behavior of the smoothness of fractional order of the conjugate functions of many variables at fixed point in the space C if the global smoothness as well as the behavior at this point of the original functions are known. The direct estimates are obtained and exactness of these estimates are established by proper examples.

Some notations and definitions

Let R^n $(n = 1, 2, \dots;\ R^1 \equiv R)$ be the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$ with real coordinates. Let B be an arbitrary non-empty subset of the set $M = \{1, \dots, n\}$. Denote by $|B|$ the cardinality of B. Let x_B be such a point in R^n whose coordinates with indices in $M \backslash B$ are zero.

As usual $C(T^n)$ ($C(T^1) \equiv C(T)$), where $T = [-\pi, \pi]$, denotes the space of all continuous functions $f : R^n \to R$ that are 2π-periodic in each variable, endowed with the norm

$$\|f\| = \max_{x \in T^n} |f(x)|.$$

Definition

If $f \in L(T^n)$, then following Zhizhiashvili we call the expression

$$\widetilde{f}_B(x) = \Big(- \frac{1}{2\pi} \Big)^{|B|} \int_{T^{|B|}} f(x + s_B) \prod_{i \in B} \cot \frac{s_i}{2} \, ds_B$$

the conjugate function of n variables with respect to those variables whose indices form the set B (with $\widetilde{f}_B \equiv \widetilde{f}$ for $n = 1$).

Suppose that $f \in C(T^n)$, $1 \le i \le n$, and $h \in R$.

Definition

For each $x \in R^n$ let us consider the difference of fractional order α ($\alpha > 0$)

$$\Delta_i^\alpha(h)\, f(x) = \sum_{j=0}^{\infty} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(x_1, \ldots, x_{i-1}, x_i + j\, h, x_{i+1}, \ldots, x_n),$$

where $\begin{pmatrix} \alpha \\ j \end{pmatrix} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$ *for* $j > 1$, $\begin{pmatrix} \alpha \\ j \end{pmatrix} = \alpha$ *for* $j = 1$, $\begin{pmatrix} \alpha \\ j \end{pmatrix} = 1$ *for* $j = 0$.

Then define the partial modulus of smoothness of order α ($\alpha > 0$) of the function f with respect to the variable x_i by the equality

$$\omega_{\alpha,i}\,(f;\delta) = \sup_{|h| \le \delta} \left\| \Delta_i^\alpha(h)\, f \right\| .$$

($\Delta_i^\alpha(h)\, f(x) \equiv \Delta^\alpha(h)\, f(x)$ and $\omega_{\alpha,i}\,(f;\delta) \equiv \omega_\alpha\,(f;\delta)$ for $n = 1$).

Let Φ_α $(\alpha > 0)$ be the set of nonnegative, continuous functions $\varphi(\delta)$ on $[0,1)$ such that

1. $\varphi(\delta) = 0$,
2. $\varphi(\delta)$ is nondecreasing,
3. $\int_0^\delta \frac{\varphi(t)}{t}dt = O(\varphi(\delta))$,
4. $\delta^\alpha \int_\delta^1 \frac{\varphi(t)}{t^{\alpha+1}}dt = O(\varphi(\delta))$.

Note that when $\alpha = k$ is an integer number then the class Φ_α coincides with the well-known class of Bari-Stechkin of order k.

Definition

Let φ be a nonnegative, nondecreasing continuous function defined on $[0,1)$ with $\varphi(\delta) = 0$. Then we denote by $H_i^\alpha(\varphi; C(T^n))$ $(i = 1, \dots, n)$ the set of all functions $f \in C(T^n)$ such that

$$\omega_{\alpha,i}(f;\delta) = O\Big(\varphi(\delta)\Big), \quad \delta \to 0+, \quad i = 1, \dots, n.$$

We set

$$H^\alpha\Big(\varphi; C(T^n)\Big) = \bigcap_{i=1}^{n} H_i^\alpha\Big(\varphi; C(T^n)\Big).$$

By I we denote the following subset of the set R^n: $\{x : x = (\underbrace{\overline{x}, \dots, \overline{x}}_{n}); \overline{x} \in T\}$.

Some known results

Moduli of smoothness play a basic role in approximation theory, Fourier analysis and their applications. For a given function f, they essentially measure the structure or smoothness of the function via the k-th difference $\Delta_i^k(h)\, f(x)$. In fact, for the functions f belonging to the Lebesgue space L^p or the space of continuous functions C, the classical k-th modulus of continuity has turned out to be a rather good measure for determining the rate of convergence of best approximation. On this direction one could see books:

[1] V. K. Dzyadyk and I. A. Shevchuk. *Theory of Uniform Approximation of Functions by Polynomials.* Walter De Gruyter, Berlin, Germany, 2008.

[2] R. M. Trigub and E. S. Belinsky. *Fourier analysis and approximation of functions.* Kluwer Academic Publishers, Dordrecht, Boston , London, 2004.

In 1977 P. L. Butzer, H. Dyckhoff, E. Goerlich, R. L. Stens and R. Tabersky introduced the modulus of smoothness of fractional order. This notion can be considered as a direct generalization of the classical modulus of smoothness and is more natural to use for a number of problems in harmonic analysis.

In the theory of real functions there is the well-known theorem of Privalov on the invariance of the Lipschitz classes under the conjugate function $\widetilde{f}$.

Namely, in 1916 Privalov proved that if the function f belongs to the class $Lip(\alpha, C(T))(0 < \alpha < 1)$ then $\widetilde{f} \in Lip(\alpha, C(T))$.

For $\alpha = 1$ Privalov's theorem is not valid.

Analogous problem in terms of modulus of smoothness of fractional order was considered by Samko and Yakubov. They proved that the generalized Holder class $H^{\alpha}\Big(\varphi; C(T)\Big)$ $(\varphi \in \Phi_{\alpha}, \alpha > 0)$ is invariant under the operator $\widetilde{f}$.

In 1924 Zygmund obtained more strong result than Privalov's theorem. Namely: If the function $f \in C(T)$ and

$$\int_0^{\pi} \frac{\omega(f;t)}{t} dt < +\infty,$$

then $\widetilde{f}$ exists everywhere, $\widetilde{f} \in C(T)$ and

$$\omega(\widetilde{f};\delta) \leq A\Big[\int_0^{\delta} \frac{\omega(f;t)}{t} dt + \delta \int_{\delta}^{\pi} \frac{\omega(f;t)}{t^2} dt\Big], 0 < \delta \leq \frac{\pi}{2}.$$

From this result we get that if the modulus of continuity ω satisfies so called Zygmund's condition then the class $H(\omega; C(T))$ is invariant under the operator $\widetilde{f}$.

In 1945 Zygmund proved that the analogous theorem of the Privalov theorem is valid for the modulus of continuity of second order in case $\alpha = 1$.

Afterwards Bari and Stechkin studied necessary and sufficient conditions for the invariance of classes $H(\varphi; C(T))$ under the conjugate operator.They showed that Zygmund's condition is exact for the invariance of $H(\omega; C(T))$ classes under the operator $\widetilde{f}$. Namely: Let ω be modulus of continuity and

$$\int_0^\pi \frac{\omega(t)}{t} dt < +\infty.$$

Then there exists the function $f \in H(\omega; C(T))$ such that

$$\omega(\widetilde{f}; \delta) \geq \frac{1}{\pi}\Big[\int_0^\delta \frac{\omega(t)}{t} dt + \delta \int_\delta^\pi \frac{\omega(t)}{t^2} dt\Big], 0 < \delta \leq \frac{\pi}{2}.$$

As to the functions of many variables, the first result in this direction belongs to Cesari and Zhak. They showed that the class $Lip(\alpha, C(T^2))(0 < \alpha < 1)$ is not invariant under the conjugate operators of two variables.Namely, in 1938 Cesari proved the following theorem: If $f \in Lip(\alpha, C(T^2))(0 < \alpha < 1)$, then

$$\omega_i(\widetilde{f}_{\{j\}}; \delta) = O(\delta^\alpha), \;\; i, j = 1, 2, \;\; \delta \to 0+,$$

$$\omega_i(\widetilde{f}_{\{1,2\}}; \delta) = O(\delta^\alpha |\ln \delta|), \;\; i = 1, 2, \;\; \delta \to 0+.$$

In his work Zhak corrected Cesari's result. He proved that if $f \in Lip(\alpha, C(T^2))(0 < \alpha < 1)$, then

$$\omega_i(\widetilde{f}_{\{i\}}; \delta) = O(\delta^\alpha), \;\; i = 1, 2, \;\; \delta \to 0+,$$

$$\omega_j(\widetilde{f}_{\{i\}}; \delta) = O(\delta^\alpha |\ln \delta|), \;\; i \neq j, i, j = 1, 2, \;\; \delta \to 0+,$$

$$\omega_i(\widetilde{f}_{\{1,2\}}; \delta) = O(\delta^\alpha |\ln \delta|), \;\; i = 1, 2, \;\; \delta \to 0+.$$

In this work the exactness of these estimates are established by proper example constructed by Landis.

Later, there were obtained the sharp estimates for partial moduli of continuity of different orders in the space of continuous functions:

[3] A. Danelia. Estimates for the modulus of continuity of conjugate functions of many variables. *East J. Approx.*, 13(1), 7-19, 2007.

[4] A. Danelia. Conjugate function and the modulus of continuity of k-th order. *Acta Math. Hungar.*, 138(3), 281-293, 2013.

[5]V. A. Okulov. A multidimensional analog of a theorem due to Zygmund. *Math. Notes*, 61(5), 600-608, 1997.

The following estimation is valid.

Theorem

a) *Let* $f \in H(\omega_k, C(T^n))$ *and for each* $B \subseteq M$

$$\int_{\left[0,\frac{\pi}{2}\right]^{|B|}} \min_{i\in B} \omega_k(s_i) \prod_{i\in B} \frac{ds_i}{s_i} < \infty.$$

Then

$$\omega_{k,j}(\widetilde{f}_B;\delta) = O\Big(\int_{[0,\pi]^{|B|}} \min(\delta^k, s_j^k) s_j^{-k} \min_{i\in B} \omega_k(s_i) \prod_{i\in B} s_i^{-1} ds_i\Big),\ \ j \in B,\ \ \delta \to 0+,$$

$$\omega_{k,j}(\widetilde{f}_B;\delta) = O\Big(\int_{[0,\pi]^{|B|}} \min\Big\{\min_{i\in B} \omega_k(s_i), \omega_k(\delta)\Big\} \prod_{i\in B} s_i^{-1} ds_i\Big),\ j \in M\setminus B,\ \ \delta \to 0+.$$

b) *For each* $B \subseteq M$ *there exist functions* F *and* G *such that* $F, G \in H(\omega_k; \mathbb{C}(T^n))$ *and*

$$\omega_{k,j}(\widetilde{F}_B;\delta) \geq C \int_{[0,\pi]^{|B|}} \min(\delta^k, s_j^k) s_j^{-k} \min_{i\in B} \omega_k(s_i) \prod_{i\in B} s_i^{-1} ds_i\ ),\ j \in B,\ \ 0 \leq \delta \leq \delta_0,$$

$$\omega_{k,j}(\widetilde{G}_B;\delta) \geq C \int_{[0,\pi]^{|B|}} \min\Big\{\min_{i\in B} \omega_k(s_i), \omega_k(\delta)\Big\} \prod_{i\in B} s_i^{-1} ds_i\ ,\ \ j \in M\setminus B,\ \ 0 \leq \delta \leq \delta_0,$$

where C *and* δ_0 *are positive constants.*

The case $k = 1$ is considered by Okulov in [5, Theorem 2, Theorem 3] and the cases $k \geq 2$ was considered by us in [3, Theorem] and [4, Theorem].

The cases when moduli of continuity of different orders satisfy Zygmund's condition were considered in works:

[6] A. Danelia. On certain properties of the conjugate functions of many variables in the spaces $C(T^m)$ and $L(T^m)$. *East J. Approx.*, 7(4), 401-415, 2001.

[7] M. M. Lekishvili and A. N. Danelia. Multidimentional conjugation operators and deformations of the classes $Z(\omega^{(2)}; \mathbb{C}(T^m))$. *Math. Notes*, 63(6), 752-759, 1998.

[8] V. A. Okulov. Multidimensional analogue of a theorem of Privalov. *Sb.Math.*, 186(2), 257-269, 1995.

Main results

In the paper
[9] A.Danelia. Conjugate Functions and the Modulus of Smoothness of Fractional Order. *Journal of Contemporary Mathematical Analysis*,53(5), 288-293, 2018
we obtained the exact estimates of the partial moduli of smoothness of fractional order of the conjugate functions of several variables in the space $H\Big(\varphi; C(T^n)\Big)$ with the condition $\varphi \in \Phi_\alpha$, $\alpha > 0$.

The following theorem is valid.

Theorem

a) Let $f \in H^\alpha(\varphi; C(T^n))$ *and* $\varphi \in \Phi_\alpha$, $\alpha > 0$. *Then*

$$\omega_{\alpha,i}(\widetilde{f}_B;\delta) = O(\varphi(\delta)|\ln\delta|^{|B|-1}),\ \ i \in B\ \delta \to 0+,$$

$$\omega_{\alpha,i}(\widetilde{f}_B;\delta) = O(\varphi(\delta)|\ln\delta|^{|B|}),\ \ i \in M\backslash B,\ \ \delta \to 0+.$$

b) For each $B \subseteq\ M$ *there exists a functions* G *such that* $G \in H\Big(\varphi; C(T^n)\Big)$ *and*

$$\omega_{\alpha,i}(\widetilde{G}_B;\delta) \ge C\varphi(\delta)|\ln\delta|^{|B|-1}\ ,\ j \in B,\ 0 \le \delta \le \delta_0,$$

$$\omega_{\alpha,i}(\widetilde{G}_B \ge C\varphi(\delta)|\ln\delta|^{|B|}\ ,\ \ j \in M \setminus B,\ 0 \le \delta \le \delta_0,$$

where C *and* δ_0 *are positive constants.*

In the following theorem we give the result on the smoothness of the conjugate functions $\widetilde{f}_B$ on the set I. If we restrict the function $\widetilde{f}_B$ on the set I, we can consider it as a function of one variable.the following question arises: what we can say about the smoothness of this "new function" if the function f belongs to $H(\varphi; C(T^n))$ and $\varphi \in \Phi_\alpha$, $\alpha > 0$.

The following theorem is valid.

Theorem

a) Let $f \in H^\alpha(\varphi; C(T^n))$ and $\varphi \in \Phi_\alpha$, $\alpha > 0$. Then

$$\sup_{h \in I, |\overline{h}| \le \delta} \sup_{x \in I} | \Delta_j^\alpha(\overline{h}) \widetilde{f}_B(x) | = O(\varphi(\delta) | \ln \delta |^{|B|-1}), \; j \in B, \; \delta \to 0+,$$

$$\sup_{h \in I, |\overline{h}| \le \delta} \sup_{x \in I} | \Delta_j^\alpha(\overline{h}) \widetilde{f}_B(x) | = O(\varphi(\delta) | \ln \delta |^{|B|}), \; j \in M \setminus B, \; \delta \to 0+ .$$

b) For each $B \subseteq M$ there exist functions F and G such that $F, G \in H^\alpha(\varphi; C(T^n))$ and

$$\sup_{h \in I, |\overline{h}| \le \delta} \sup_{x \in I} | \Delta_j^\alpha(\overline{h}) \widetilde{F}_B(x) | \ge C\varphi(\delta) | \ln \delta |^{|B|-1}, \; j \in B, \; 0 \le \delta \le \delta_0,$$

$$\sup_{h \in I, |\overline{h}| \le \delta} \sup_{x \in I} | \Delta_j^\alpha(\overline{h}) \widetilde{G}_B(x) | \ge C\varphi(\delta) | \ln \delta |^{|B|} , \; j \in M \setminus B, \; 0 \le \delta \le \delta_0,$$

where C and δ_0 are positive constants.

Note that the analogous result is given in the work
[10] A.Danelia. On some local properties of the conjugate function and the modulus of continuity of k-th order. *Acta Mathematica Academiae Paedagogicae Nygyhensis*, 33(1), 45-51, 2017
when the function f belongs to $H(\omega_k; C(T^n))$ and the modulus of continuity ω_k satisfies Zygmund's condition.

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Thank you for your attention!