Bounds for the sharp constant in the Jackson–Nikol'skii inequality between the uniform norm and integral norm of trigonometric polynomials

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Abstarct

The talk will contain the results improving known two-sided bounds for the sharp constant in the Jackson–Nikol'skii inequality between the uniform norm and integral norm of trigonometric polynomials obtained by L.V. Taikov and D.V. Gorbachev earlier.
We extend an approach proposed by Ya.L. Geronimus and F. Peherstorfer to characterize canonical sets of points in the space of trigonometric polynomials on the period in the case of the unit weight to the case of a special Jacobi weight.

Statement of the problem

Denote by $\mathcal{T}_n$ the subspace of real-valued trigonometric polynomials

$$\tau(t) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu t + b_\nu \sin \nu t) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu t}, \quad a_\nu, b_\nu \in \mathbb{R}.$$

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} = (-\pi, \pi]$ be the period of length 2π.
D. Jackson (1933) proved the inequality

$$\|\tau\|_C \leq 2n\|\tau\|_L, \qquad \tau \in \mathcal{T}_n,$$

where

$$\|\tau\|_C = \max_{t \in \mathbb{T}} |\tau(t)|, \quad \|\tau\|_L = \int_{\mathbb{T}} |\tau(t)|\, dt.$$

We are interested in an exact constant γ_n in the inequality

$$\|\tau\|_C \le \gamma_n \|\tau\|_L, \quad \tau \in \mathcal{T}_n,$$

i.e., in the value

$$\gamma_n = \sup_{\tau \in \mathcal{T}_n,\, \tau \not\equiv 0} \frac{\|\tau\|_C}{\|\tau\|_L}.$$

In 1930th, Ya.L. Geronimus and N.I. Akhiezer with M.G. Krein established that the problem on calculating this value reduces to finding the largest root of a special equation written in terms of the determinant of $n \times n$-matrix satisfying special conditions. However, this result does not show the behavior of γ_n as $n \to \infty$.

History

Studying this problem in his paper of 1965, L.V. Taikov refers to S.B. Stechkin's result about the fact that there exists a finite positive constant γ such that

$$\gamma_n = \gamma n + o(n) \quad \text{as} \quad n \to \infty.$$

Jackson's result (1933) mentioned above implies the upper bound

$$\gamma \leq 2.$$

Taikov (1965) proved the bounds

$$\frac{1.07995\ldots}{2\pi} \leq \gamma \leq \frac{1.16625\ldots}{2\pi}.$$

D.V. Gorbachev (2003) refined these bounds:

$$\frac{1.08185\ldots}{2\pi} \leq \gamma \leq \frac{1.09769\ldots}{2\pi}.$$

In addition, he proved that the values

$$\nu = 2\pi\gamma, \quad \nu_n = 2\pi\gamma_n$$

are related by the inequalities

$$n\nu \leq \nu_n \leq (n+1)\nu. \tag{1}$$

We also should note the following result by V.F. Babenko, V.A. Kofanov, and S.A. Pichugov (2002):

$$\sup_{n\in\mathbb{N}} \frac{\nu_{n+1}}{n} = \nu_2 = \frac{\pi}{2\xi} = 2.12532\ldots,$$

where ξ is the unique root of the equation $\cos t = t$. This result implies the inequality $\nu_n \leq (n-1)\nu_2$ $(n = 2, 3, \ldots)$, which for $n > 3$ is weaker than the former inequality in (1).

Denote by $\mathcal{C}_n$ the subspace of even trigonometric polynomials $\tau \in \mathcal{T}_n$; i.e., $\mathcal{C}_n$ is the subspace of cosine-polynomials

$$\tau(t) = \sum_{\nu=0}^{n} a_\nu \cos \nu t$$

of order at most n with real coefficients $a_0, a_1, \dots, a_n$. Denote by $\mathbf{a} = \mathbf{a}(\tau)$ the vector composed from coefficients of a cosine-polynomial τ:

$$\mathbf{a} = \mathbf{a}(\tau) = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}.$$

The subspace $\mathcal{T}_n$ and the norms $\|\cdot\|_C$ and $\|\cdot\|_L$ are shift invariant. In view of this fact and other known reasonings, it is easy to show that the following chain of equalities holds:

$$\gamma_n = \sup_{\tau\in\mathcal{T}_n,\tau\not\equiv 0} \frac{\|\tau\|_C}{\|\tau\|_L} = \sup_{\tau\in\mathcal{T}_n,\tau\not\equiv 0} \frac{|\tau(0)|}{\|\tau\|_L} = \sup_{\tau\in\mathcal{T}_n,\tau\not\equiv 0} \frac{\tau(0)}{\|\tau\|_L} = \sup_{\tau\in\mathcal{C}_n,\tau\not\equiv 0} \frac{\tau(0)}{\|\tau\|_L}.$$

It is clear that the following equality is valid:

$$\gamma_n = \frac{1}{\mu_n},$$

in which

$$\mu_n = \inf\{\|\tau\|_L \colon \tau \in \mathcal{C}_n,\ \tau(0) = 1\}.$$

It is also obvious that the value μ_n can be represented in the form

$$\mu_n = \inf\{\|\tau\|_L \colon \tau \in \mathcal{C}_n,\ (\mathbf{1}, \mathbf{a}(\tau)) = 1\},$$

where $(\mathbf{1}, \mathbf{a}) = \sum_{\nu=0}^{n} a_\nu$ is the usual inner product of the vectors $\mathbf{1} = (1, 1, \ldots, 1)$ and $\mathbf{a} = (a_0, a_1, \ldots, a_n)$ in $\mathbb{R}^{n+1}$. Using this representation and one of V. Markov's ideas (1892), we reduce the problem on μ_n to finding the value of the best integral approximation on $(0, \pi)$ to the constant function equal identically to 1 by polynomials in terms of the system of functions

$$\Phi_n = \{1 - \cos t, (1 - \cos t) \cos t, (1 - \cos t) \cos 2t, \ldots, (1 - \cos t) \cos (n-1)t\}.$$

More precisely,

$$\mu_n = 2 \inf_{\tau \in \mathcal{C}_{n-1}} \|1 - (1 - \cos t)\tau(t)\|_{L(0,\pi)}, \tag{2}$$

where $\|f\|_{L(0,\pi)} = \int_0^\pi |f(t)|\,dt$. Here, we used the fact that $\|f\|_L = 2\|f\|_{L(0,\pi)}$ for even 2π-periodic functions f. In other words, we have the equalities

$$\mu_n = 2E(1, \operatorname{span}\Phi_n, L(0,\pi)) = 2E(1, w \cdot \mathcal{C}_{n-1}, L(0,\pi)),$$

where

$$w(t) = 1 - \cos t$$

and $E(f, Y, X)$ is the value of the best approximation of an element f of a normed space X by a subspace $Y \subset X$.

Since Φ_n is a Chebyshev system of functions in the interval $(0, \pi)$ and the approximated function is continuous in this interval, we conclude that, by Jackson's theorem, there exists a unique polynomial

$$g^* \in \mathrm{span}\,\Phi_\mathrm{n}$$

of the best integral approximation; i.e., the infimum on the right-hand side of equality (2) is attained at the unique polynomial

$$g^*(t) = (1 - \cos t)\tau^*(t), \quad \text{where} \quad \tau^* \in \mathcal{C}_{n-1}.$$

In addition, if we add the function identically equal to 1 to the system Φ_n, then the extended system is also Chebyshev in the interval $(0, \pi)$. Therefore, arguing similar to S.M. Nikol'slii (1947) and applying a result by S.N. Bernstein (1937), we conclude that this extremal polynomial g^* interpolates the approximated constant function at exactly n points $t_1 < t_2 < \cdots < t_n$ located in the interval $(0, \pi)$, which coincide with zeros of the cosine-polynomial τ_n of order n with unit leading coefficient

$$\tau_n(t) = \cos nt + \sum_{k=0}^{n-1} \widehat{a}_k \cos kt$$

that deviates least from zero in the following sense:

$$\inf_{\tau \in \mathcal{C}_{n-1}} \int_0^{\pi} \left| \cos nt - \tau(t) \right| (1 - \cos t)\, dt = \int_0^{\pi} \left| \tau_n(t) \right| (1 - \cos t)\, dt. \quad (3)$$

Applying the duality relations to the right-hand side of (2), we come to the equalities

$$\frac{\mu_n}{2} = \sup_{F \perp w \cdot C_{n-1},\ \|F\|_{L_\infty} \le 1} \int_0^\pi F(t)\,dt, \quad n \in \mathbb{N}. \tag{4}$$

Moreover, there exists a function $F_0 \in L_\infty$ such that $F_0 \perp w \cdot \mathcal{C}_{n-1}$, $\|F_0\|_{L_\infty} = 1$, and at which the supremum in (4) is attained. Here, $F \perp w \cdot \mathcal{C}_{n-1}$ means that

$$\int_0^\pi F(t)\tau(t)w(t)\,dt = 0 \quad \text{for any} \quad \tau \in \mathcal{C}_{n-1}.$$

In addition, the extremal function F_0 is unique; moreover, it is a signum-function, i.e., $|F_0(t)| = 1$ at all points t from $(0, \pi)$ except for set of points of sign variation $t_1 < t_2 < \cdots < t_n$ from $(0, \pi)$. This family of points $\{t_1, t_2, \ldots, t_n\} \subset (0, \pi)$ coincides with the family of zeros of the cosine-polynomial τ_n of order n with the unit leading coefficient that deviates least from zero.

To construct the extremal signum-function F_0, we use a finite Blaschke product defined by algebraic polynomials with real coefficients, all zeros of which lie inside the unit disk of the complex plane. We apply results by Ya.L. Geronimus and F. Peherstorfer as well as modifications of their ideas. This made it possible to reduce the problem under consideration for a fixed n to finding the root of an equation involving both algebraic and trigonometric functions. Due to bounds (1) established by Gorbachev (2003), we succeeded in good localization of the required root and finding it numerically with any accuracy using the analytic software Maple.
In turn, this made it possible to improve the upper bound for γ. Thus, we have

$$\frac{1.081}{2\pi} \le \gamma \le \frac{1.082}{2\pi}.$$

Note that Hörmander and Bernhardsson (1993), by other methods, showed that

$$\gamma \approx \frac{1.08185}{2\pi}.$$

The problem of calculating γ remains open.
The importance of estimating constants similar to γ in the theory of the Riemann zeta function was noted by E. Carneiro, M.B. Milinovich, and K. Soundararajan (2018).

Example: $n = 4$

The problem of calculating γ_n for $n = 4$ reduces to finding the root $\lambda = \lambda(4)$ of the equation $\mathcal{N}(\lambda) = 0$ in the interval $\lambda \in (-0.1815,\, 0.1815)$, where

$$\mathcal{N}(\lambda) = -180\,\lambda^2 \cos(\lambda) + 288\,\cos(\lambda)\,\lambda^3 - 96\,\cos(\lambda)\,\lambda^5 - 96\,\cos(\lambda)\,\lambda^4 +$$

$$+64\,\cos(\lambda)\,\lambda^6 - 36\,\lambda\,\cos(\lambda) - 288\,(\cos(\lambda))^3\,\lambda^3 - 96\,(\cos(\lambda))^3\,\lambda^4 - 9\,\cos(\lambda) +$$

$$+9\,\sin(\lambda) - 72\,\sin(\lambda)\,\lambda\,(\cos(\lambda))^2 + 288\,\sin(\lambda)\,\lambda^2\,(\cos(\lambda))^2 - 64\,\sin(\lambda)\,\lambda^6 -$$

$$-96\,\sin(\lambda)\,\lambda^5 + 192\,\sin(\lambda)\,\lambda^4 - 108\,\sin(\lambda)\,\lambda^2 + 36\,\sin(\lambda)\,\lambda + 288\,\lambda^2\,(\cos(\lambda))^3$$

$$+72\,\lambda\,(\cos(\lambda))^3 - 36\,\sin(\lambda)\,(\cos(\lambda))^2 - 36\,(\cos(\lambda))^5 + 36\,\sin(\lambda)\,(\cos(\lambda))^4 -$$

$$-96\,\sin(\lambda)\,(\cos(\lambda))^2\,\lambda^4 + 36\,(\cos(\lambda))^3 + 288\,\sin(\lambda)\,\lambda^3\,(\cos(\lambda))^2,$$

Example: $n = 4$

The numerical value of the solution is

$$\lambda(4) = 0.1456010855826293787590220108 43\ldots .$$

Hence,

$$c(4) = \frac{\pi}{|4 \cdot 4 \cdot \lambda(4)|} = 1.3485444841544995517861239538 2\ldots ,$$

where

$$c(n) = \frac{2\pi\gamma_n}{n}.$$

Thank you for your attention!